Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Systems Research Institute
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On $\max -*$ fuzzy systems

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Abstract

In this article we investigate problem of solving fuzzy linear equations for $\max -*$ product. The problem of solving fuzzy relational equations and inequalities with $\max - \min$ product was introduced by Sanchez in 1976 and by Pappis and Sugeno in 1985.

Our considerations concern solving fuzzy linear equations with $\max -*$ product, where $*$ is a binary operation with additional properties. We illustrate this problem for different classes of binary operation $*$. We investigate correlation between properties of the binary operation and the form and properties of solution such equations.

Moreover, we illustrate the problem of solving of fuzzy linear systems based on solving fuzzy linear equations with $\max - \min$ product.

Keywords: Fuzzy equations, fuzzy relation equations, systems of equations, systems of $\max -*$ equations.

1 Introduction

In this paper we present properties of systems of $\max -*$ equations, where $*$ is an operation with additional properties. We present relationship between assumptions on a binary operation $*$ and $\max -*$ equations. We study correlation between assumption on the binary operation $*$ and set of solution fuzzy linear equations and inequalities.

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In the second part we present correlation between different classes of equations. Moreover we determine the set of solution fuzzy linear equations by determination of the set of fuzzy linear equations.

For short, in this article operations $\vee$ and $\wedge$ are used for numbers:

$$x \vee y = \max\{x, y\}, \quad x \wedge y = \min\{x, y\}, \quad x, y \in [0,1] \tag{1}$$

$$\bigvee_{t\in T} x_t = \max_{t\in T}\{x_t\}, \quad \bigwedge_{t\in T} x_t = \min_{t\in T}\{x_t\}, \quad x_t \in [0,1],\ T \neq \emptyset. \tag{2}$$

We use the following order

$$(B \leqslant C) \Leftrightarrow (b_{ij} \leqslant c_{ij}, i \in \{1,\dots,m\}, j \in \{1,\dots,p\}), B, C \in [0,1]^{m\times p}. \tag{3}$$

Definition 1. By interval in $[0,1]^{m\times p}$ we call

$$[C, D] = \{X \in [0,1]^{m\times p} : C \leqslant X \leqslant D\},\ C \leqslant D,\ \ C, D \in [0,1]^{m\times p}. \tag{4}$$

Corollary 1. *For not disjoint intervals in $[a,b],[c,d] \in [0,1]$, we get intersection*

$$[a,b] \cap [c,d] = [a \vee c, b \wedge d], \tag{5}$$

where $a \leqslant b$ and $c \leqslant d$.

Example 1. Let $A, B \in [C, D]$, where

$$A = \begin{bmatrix} 0.3 & 0.7 \\ 0.2 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.1 & 0.6 \\ 0 & 0.2 \end{bmatrix}, D = \begin{bmatrix} 0.5 & 0.7 \\ 1 & 0.9 \end{bmatrix}.$$

We observe that A and B can be incomparable.

Definition 2 (cf. [1], Definition 11)**.** An operation $* : [0,1]^2 \to [0,1]$ is called infinitely $\sup -$ distributive ($\inf -$ distributive) if

$$\underset{a,b_t\in[0,1]}{\forall} \ a * (\bigvee_{t\in T} b_t) = \bigvee_{t\in T}(a * b_t), \quad (\bigvee_{t\in T} b_t) * a = \bigvee_{t\in T}(b_t * a) \tag{6}$$

$$\left(\underset{a,b_t\in[0,1]}{\forall} \ a * (\bigwedge_{t\in T} b_t) = \bigwedge_{t\in T}(a * b_t), \quad (\bigwedge_{t\in T} b_t) * a = \bigwedge_{t\in T}(b_t * a) \right) \tag{7}$$

for arbitrary index set $T \neq \emptyset$ and any $a, b_t \in [0,1]$ for $t \in T$.

Lemma 1 (cf. [3], Proposition 1.22)**.** *Let an operation* $* : [0,1]^2 \to [0,1]$ *be increasing. The operation* $*$ *is left continuous if and only if the operation* $*$ *is infinitely* $\sup -$ *distributive.*

Definition 3 (cf. [8])**.** Let $* : [0,1]^2 \to [0,1]$. The matrix $A \circ B$ is called $\max -*$ product of matrices $A \in [0,1]^{m\times n}$ and $B \in [0,1]^{n\times p}$ if

$$(A \circ B)_{ik} = \bigvee_{j=1}^{n} (a_{ij} * b_{jk}), \quad i = 1,2,\ldots,m, \; k = 1,2,\ldots,p. \tag{8}$$

Example 2. Let $* = \wedge$ and

$$A = \begin{bmatrix} 0.3 & 0.6 & 0.3 \\ 0.7 & 1 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0.2 \\ 0.5 & 1 \\ 0.7 & 0.6 \end{bmatrix}.$$

We get

$$A \circ B = \begin{bmatrix} 0.5 & 0.6 \\ 0.7 & 1 \end{bmatrix}, \quad B \circ A = \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 1 & 0.4 \\ 0.6 & 0.6 & 0.4 \end{bmatrix}.$$

Thus product (8) is not commutative.

Lemma 2 (cf. [2], Lemma 3)**.** *If an operation* $* : [0,1]^2 \to [0,1]$ *is increasing, then* $\max -*$ *product is isotonic, i.e.*

$$(A \leqslant C) \Rightarrow (A \circ x \leqslant C \circ x), \quad (x \leqslant y) \Rightarrow (A \circ x \leqslant A \circ y), \tag{9}$$

for $A, C \in [0,1]^{m\times n}$, $x, y \in [0,1]^n$.

2 Fuzzy linear equations

Let $* : [0,1]^2 \to [0,1]$. In this part we analyze properties of sets $L(a,b,*)$, $U(a,b,*)$ and $E(a,b,*)$ for $a, b \in [0,1]$, where

$$L(a,b,*) = \{t \in [0,1] : a * t \leqslant b\}, \tag{10}$$

$$E(a,b,*) = \{t \in [0,1] : a * t = b\}, \tag{11}$$

$$U(a,b,*) = \{t \in [0,1] : a * t \geqslant b\}. \tag{12}$$

We have

$$E(a,b,*) = L(a,b,*) \cap U(a,b,*). \tag{13}$$

Example 3. For $a, b \in (0,1)$ and the following operation

$$x * y = \begin{cases} 1 & \text{for } y \in [0,1] \cap \mathbb{Q} \\ 0 & \text{for } y \in [0,1] \backslash \mathbb{Q} \end{cases} \quad , x, y \in [0,1]$$

we have $U(a,b,*) = [0,1] \cap \mathbb{Q}$ and $L(a,b,*) = [0,1] \backslash \mathbb{Q}$. Therefore, the sets $L(a,b,*)$ and $U(a,b,*)$ are dense subsets of $[0,1]$ and they are disjoint.

Example 4. Sets $L(a,b,*)$ and $U(a,b,*)$ can be interwoven even for continuous operation $*$. If

$$x * y = |x - y|, \quad x, y \in [0,1],$$

then

$$L\left(\frac{1}{2}, \frac{1}{4}, *\right) = \left[\frac{1}{4}, \frac{3}{4}\right], \; U\left(\frac{1}{2}, \frac{1}{4}, *\right) = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right],$$

$$E\left(\frac{1}{2}, \frac{1}{4}, *\right) = \left\{\frac{1}{4}, \frac{3}{4}\right\}.$$

For the following operation

$$x * y = \begin{cases} \frac{1}{2}\left(1 + \sin\left(\frac{y}{x}\right)\right) & , x > 0 \\ 1 & , x = 0 \end{cases} \quad , \quad x, y \in [0,1],$$

the sets $L(a,b,*)$, $U(a,b,*)$ will consist of several separate intervals. It depends on a and b. For example we have

$$L\left(\frac{1}{7\Pi}, \frac{1}{2}, *\right) = \left[\frac{1}{7}, \frac{2}{7}\right] \cup \left[\frac{3}{7}, \frac{4}{7}\right] \cup \left[\frac{5}{7}, \frac{6}{7}\right],$$

$$U\left(\frac{1}{7\Pi}, \frac{1}{2}, *\right) = \left[0, \frac{1}{7}\right] \cup \left[\frac{2}{7}, \frac{3}{7}\right] \cup \left[\frac{4}{7}, \frac{5}{7}\right] \cup \left[\frac{6}{7}, 1\right],$$

$$E\left(\frac{1}{7\Pi}, \frac{1}{2}, *\right) = \left\{0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right\}.$$

Example 5. Let us consider monotonic operations. If $*$ is a constant operation. For example, when

$$x * y = \frac{1}{2}, \quad x, y \in [0,1],$$

then we obtain $L(a,b,*) = [0,1]$, $U(a,b,*) = \emptyset$ and $E(a,b,*) = \emptyset$ for $b > \frac{1}{2}$, $L(a,b,*) = U(a,b,*) = E(a,b,*) = [0,1]$ for $b = \frac{1}{2}$, $L(a,b,*) = \emptyset$, $U(a,b,*) = [0,1]$, $E(a,b,*) = \emptyset$ for $b < \frac{1}{2}$. If

$$x * y = \frac{x + y}{2}, \quad x, y \in [0,1],$$

then we have similar results, that is $L(a, b, *) = [0, 1]$, $U(a, b, *) = \emptyset$, $E(a, b, *) = \emptyset$ for $2b - a > 1$, $L(a, b, *) = [0, 2b - a]$, $U(a, b, *) = [2b - a, 1]$, $E(a, b, *) = \{2b - a\}$ for $0 \leqslant 2b - a \leqslant 1$, $L(a, b, *) = \emptyset$, $U(a, b, *) = [0, 1]$, $E(a, b, *) = \emptyset$ for $2b < a$.

This examples, we notice that the sets $L(a, b, *)$ and $U(a, b, *)$ for monotonic operations are empty sets or consistent intervals in $[0, 1]$.

We discuss non-emptiness of lower and upper sets.

Lemma 3. *Let $*$ be a binary operation in $[0, 1]$. If $x * y \leqslant x$ for $x, y \in [0, 1]$, then $L(a, b, *) = [0, 1]$ for $a \leqslant b$. If $x * y \leqslant y$ for $x, y \in [0, 1]$, then $[0, b] \subset L(a, b, *)$. In particular, when $x * 0 = 0$ for $x \in [0, 1]$, then $L(a, b, *) \neq \emptyset$.*

Proof. When $a * t \leqslant a$ for $t \in [0, 1]$ and $a \leqslant b$, then we obtain $L(a, b, *) = [0, 1]$. When $a * t \leqslant t \leqslant b$, then $[0, b] \subset L(a, b, *)$. In particular, we have $a * 0 \leqslant 0$, hence $0 \in L(a, b, *)$. It means $L(a, b, *) \neq \emptyset$. $\square$

In a consequence of assumption of this lemma we have

Corollary 2. *If $* \leqslant \wedge$, then $L(a, b, *) = [0, 1]$ for $a \leqslant b$ and $[0, b] \subset L(a, b, *)$ for $a > b$.*

In a dual way we get

Lemma 4. *Let $*$ be a binary operation in $[0, 1]$. If $x * y \geqslant x$ for $x, y \in [0, 1]$, then $U(a, b, *) = [0, 1]$ for $a \geqslant b$. If $x * y \geqslant y$ for $x, y \in [0, 1]$, then $[b, 1] \subset U(a, b, *)$.*

Corollary 3. *If $* \geqslant \vee$, then $U(a, b, *) = [0, 1]$ for $a \geqslant b$ and $[b, 1] \subset U(a, b, *)$ for $a < b$.*

Lemma 5. *Let $*$ be a binary operation in $[0, 1]$. If the operation $*$ is increasing and it has neutral element $e = 1$, then $U(a, b, *) \neq \emptyset$ for $a \geqslant b$ and $U(a, b, *) = \emptyset$ for $a < b$.*

Proof. Let $a \geqslant b$. Because the operation $*$ has neutral element $e = 1$, we get $a * 1 = a \geqslant b$. It means $1 \in U(a, b, *)$. Hence, we have $U(a, b, *) \neq \emptyset$ for $a \geqslant b$. Let $a < b$. From the assumption that the operation $*$ is increasing we get

$$a * t \leqslant a * 1 = a < b \quad \text{for } \ t \in [0, 1].$$

It means $U(a, b, *) = \emptyset$. $\square$

Since we get

Corollary 4. *If an increasing operation $*$ has neutral element $e = 1$, then we have*

$$E(a, b, *) \neq \emptyset \Rightarrow a \geqslant b. \tag{14}$$

Proof. Let suppose that $a < b$. Based on (13) and from Lemma 5 we have

$$E(a, b, *) = L(a, b, *) \cap U(a, b, *) = L(a, b, *) \cap \emptyset = \emptyset,$$

it means if $E(a, b, *) \neq \emptyset$, then $a \geqslant b$. $\square$

Example 6. Let $a = 0.6$, $b = 0.4$ and

$$x * y = \begin{cases} x \cdot y & \text{for } x, y < 0.5 \\ x \wedge y & \text{else} \end{cases}, x, y \in [0, 1].$$

We obtain the following sets $L(0.6, 0.4, *) = [0, 0.5)$, $U(0.6, 0.4, *) = [0.5, 1]$ and $E(0.6, 0.4, *) = \emptyset$. Therefore the implication in (14) cannot be replaced with equivalence. Moreover, for non-empty sets $L(a, b, *)$ and $U(a, b, *)$ we can obtain empty set $E(a, b, *)$.

We prove that monotonicity of operation $*$ suffice for connectedness of sets $L(a, b, *)$, $U(a, b, *)$ and $E(a, b, *)$

Lemma 6. *Let $*$ be a binary operation in $[0, 1]$. If the operation $*$ is increasing, then*

$$x \in L(a, b, *) \Leftrightarrow [0, x] \subset L(a, b, *) \quad \textit{for } x \in [0, 1], \tag{15}$$

$$y \in U(a, b, *) \Leftrightarrow [y, 1] \subset U(a, b, *) \quad \textit{for } y \in [0, 1]. \tag{16}$$

Proof. Let $x \in L(a, b, *)$. Because $*$ is an increasing operation, then

$$a * t \leqslant a * x \leqslant b \quad \text{for} \quad t \leqslant x.$$

Hence, we have $[0, x] \subset L(a, b, *)$.

Let $[0, x] \subset L(a, b, *)$. In particular, we get $x \in L(a, b, *)$. It proves (15). In the similar way we prove (16). $\square$

As a consequence of Lemma 6 we have

Corollary 5. *If the operation $*$ is increasing, then*

$$(y \in U(a, b, *),\ x \in L(a, b, *) \textit{ and } y \leqslant x) \Leftrightarrow [y, x] \subset E(a, b, *). \tag{17}$$

Corollary 6. *If the operation $*$ is increasing, then $L(a, b, *)$ and $U(a, b, *)$ are convex sets (intervals). In particular, $L(a, b, *)$ and $U(a, b, *)$ can be empty sets.*

Example 7. Let $x * y = |1 - x - y|$ for $x, y \in [0, 1]$ and $a = 0.6$, $b = 0.4$. We get $E(0.6, 0.4, *) = \{0, 0.8\}$, $L(0.6, 0.4, *) = \{0\} \cup [0.4, 0.8]$ and $U(0.6, 0.4, *) = [0, 0.4] \cup [0.8, 1]$. Since the operation $*$ is not increasing, $L(0.6, 0.4, *)$ and $U(0.6, 0.4, *)$ are not convex sets, we see that assumption on monotonicity in Corollary 6 is important.

Based on Example 6, the sets $L(a, b, *)$, $U(a, b, *)$ and $E(a, b, *)$ can be open (or right-open, left-open) intervals. We ask when the sets $U(a, b, *) \neq \emptyset$ and $L(a, b, *) \neq \emptyset$ are closed intervals.

Theorem 1. *Let a binary operation $*$ be increasing, $a, b \in [0, 1]$ and $L(a, b, *) \neq \emptyset$. If the operation $*$ is left continuous, then the sets $L(a, b, *)$ has the greatest element. It means that $L(a, b, *) = [0, \max L(a, b, *)]$.*

Proof. The least upper bound $p = \sup L(a, b, *)$ can be approximated by an increasing sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in L(a, b, *)$. From Lemma 1 we obtain

$$a * p = a * \bigvee_{n\in\mathbb{N}} x_n = \bigvee_{n\in\mathbb{N}} (a * x_n) \leqslant \bigvee_{n\in\mathbb{N}} b = b.$$

Since $p \in L(a, b, *)$ we get $p = \max L(a, b, *)$. □

In the similar way we get

Theorem 2. *Let a binary operation $*$ be increasing, $a, b \in [0, 1]$ and $U(a, b, *) \neq \emptyset$. If the operation $*$ is right continuous, then the set $U(a, b, *)$ has the least element. It means that $U(a, b, *) = [\min U(a, b, *), 1]$.*

From the previous theorems and Corollary 5 we obtain

Theorem 3. *If a binary operation $*$ is increasing, continuous, $a, b \in [0, 1]$ and $\min U(a, b, *) \leqslant \max L(a, b, *)$, then we have*

$$E(a, b, *) = [\min U(a, b, *), \max L(a, b, *)]. \tag{18}$$

Theorem 4. *Let a binary operation $*$ be increasing, continuous operation, $a, b \in [0, 1]$ and $\min U(a, b, *) \leqslant \max L(a, b, *)$ and $b \neq 0$. If $E(a, b, *) \neq \emptyset$ and the operation $*$ satisfies conditional cancellation law, i.e.*

$$(a * x = a * y > 0) \Rightarrow (x = y), \qquad a, x, y \in [0, 1], \tag{19}$$

then the set $E(a, b, *)$ is a singleton.

Proof. Let $x, y \in E(a, b, *)$. Because the operation $*$ satisfies conditional cancellation law, we have

$$a * x = a * y = b > 0 \Rightarrow x = y.$$

From Theorem 3 we obtain $\max L(a, b, *) = \min U(a, b, *)$ in (18) i.e. the set $E(a, b, *) = \{\max L(a, b, *)\}$. □

3 Discussion of fuzzy systems

Let consider dependences between different classes of $\max -*$ equations. This topic can be divided on the following related parts:
• fuzzy relation equations (FRE),
• fuzzy matrix equations (FME),
• fuzzy system of equations (FSE),
• fuzzy linear equations (FLE).

Let consider FRE in the following form:

$$R \circ X = T, \tag{20}$$

where are given fuzzy relations $R : Y \times Z \to [0, 1]$, $T : Y \times V \to [0, 1]$ and $X : Z \times V \to [0, 1]$ is unknown relation. Because fuzzy relation R on the finite sets can be written in the matrix form, then system FRE on finite set $Y, Z, V \neq \emptyset$ can be considered as FME:

$$A \circ X = B, \tag{21}$$

where $A \in [0, 1]^{m \times n}$ and $B \in [0, 1]^{m \times p}$ are matrix representation of the relations R and T, and $X \in [0, 1]^{n \times p}$ is matrix representation of unknown relation. However, FRE is more general issue than FME, because relational composition on infinite set requires additional assumptions (cf. [1]). Therefore FME are special case of FRE.

Let us consider FME of the form (21). Denote column vectors of matrix X and B by x^i, b^i for $1 \leqslant i \leqslant p$. Note that equation (21) can be divided on p equations (cf. (8))

$$A \circ x^1 = b^1, \quad A \circ x^2 = b^2, \quad \ldots, \quad A \circ x^p = b^p, \tag{22}$$

Therefore solving FME can be considered as solving p system of equations in the form (22). Especially, for $p = 1$ in (22) we obtain that FSE

$$A \circ x = b, \tag{23}$$

are special case of FME, where we know the matrix $A \in [0,1]^{m\times n}$ and the vector $b \in [0,1]^m$ and the vector $x \in [0,1]^n$ is unknown.

Let consider relationship between FSE an FLE. Let denote the row-vector of the matrix A by a^k for $1 \leqslant k \leqslant m$. Equation (23) can be written as the system of equations FLE

$$a^1 \circ x = b_1, \quad a^2 \circ x = b_2, \quad \ldots, \quad a^m \circ x = b_m, \tag{24}$$

Hence for $m = 1$, $a = a^1$, $c = b_1$ in (24), it means

$$a \circ x = c, \tag{25}$$

we get that FLE are special case of FSE.

Let us consider a few examples for $* = \wedge$ in the same way it was done in the monograph [6].

Example 8. Let $c = 0.5$ and

$$a^1 = \begin{bmatrix} 0.5 & 0.3 & 0.7 \end{bmatrix}.$$

We consider equation $a^1 \circ x = c$, i.e.

$$(0.5 \wedge x_1) \vee (0.3 \wedge x_2) \vee (0.7 \wedge x_3) = 0.5.$$

We have
- $0.5 \wedge x_1 \leqslant 0.5$ for $x_1 \in [0,1]$ and $0.5 \wedge x_1 = 0.5$ for $x_1 \in [0.5,1]$,
- $0.3 \wedge x_2 \leqslant 0.3 < 0.5$ for $x_2 \in [0,1]$,
- $0.7 \wedge x_3 \leqslant 0.5$ for $x_3 \in [0,0.5]$, $0.7 \wedge x_3 = 0.5$ for $x_3 = 0.5$ and $0.7 \wedge x_3 > 0.5$ for $x_3 \in (0.5,1]$.

If $x_1 \in [0.5,1]$, we can choose arbitrary $x_2 \in [0,1]$ and $x_3 \in [0,0.5]$, i.e.

$$x \in [0.5,1] \times [0,1] \times [0,0.5].$$

If $x_3 = 0.5$, we can choose arbitrary $x_1 \in [0,1]$ and $x_2 \in [0,1]$, i.e.

$$x \in [0,1] \times [0,1] \times \{0.5\}.$$

Putting

$$v^1 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}, \quad w^1 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, \quad u^1 = \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix},$$

solution set can be written in the form

$$S_1 = [v^1, u^1] \cup [w^1, u^1],$$

where $[v^1, u^1]$ and $[w^1, u^1]$ are intervals in $[0,1]^3$.

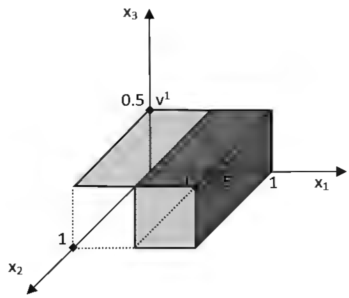

Figure 1: Solution set of $a^1 \circ x = c$.

Example 9. Let $c = 0.5$ and

$$a^2 = \begin{bmatrix} 0.5 & 0.6 & 0.8 \end{bmatrix}.$$

We consider equation $a^2 \circ x = c$, i.e.

$$(0.5 \wedge x_1) \vee (0.6 \wedge x_2) \vee (0.8 \wedge x_3) = 0.5.$$

We have
• $0.5 \wedge x_1 \leqslant 0.5$ for $x_1 \in [0,1]$ and $0.5 \wedge x_1 = 0.5$ for $x_1 \in [0.5,1]$,
• $0.6 \wedge x_2 \leqslant 0.5$ for $x_2 \in [0,0.5]$, $0.6 \wedge x_2 = 0.5$ for $x_2 = 0.5$ and $0.6 \wedge x_2 > 0.5$ for $x_2 \in (0.5,1]$,
• $0.8 \wedge x_3 \leqslant 0.5$ for $x_3 \in [0,0.5]$, $0.8 \wedge x_3 = 0.5$ for $x_3 = 0.5$ and $0.8 \wedge x_3 > 0.5$ for $x_3 \in (0.5,1]$.
If $x_1 \in [0.5,1]$, we can choose arbitrary $x_2 \in [0,0.5]$ and $x_3 \in [0,0.5]$, it means

$$x \in [0.5,1] \times [0,0.5] \times [0,0.5]. \tag{26}$$

If $x_2 = 0.5$, we can choose arbitrary $x_1 \in [0,1]$ and $x_3 \in [0,0.5]$, i.e.

$$x \in [0,1] \times \{0.5\} \times [0,0.5]. \tag{27}$$

If $x_3 = 0.5$, we can choose arbitrary $x_1 \in [0,1]$ and $x_2 \in [0,0.5]$, i.e.

$$x \in [0,1] \times [0,0.5] \times \{0.5\}. \tag{28}$$

Denote

$$v^2 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, \quad w^2 = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix}, \quad y^2 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}, \quad u^2 = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

Now, set of solutions from (26), (27) and (28) can be written in the form

$$S_2 = [v^2, u^2] \cup [w^2, u^2] \cup [y^2, u^2]$$

with relevant lattice intervals.

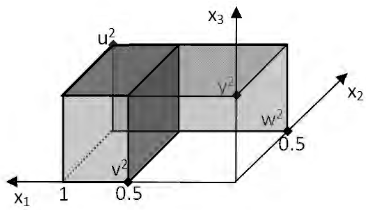

Figure 2: Set of solutions $a^2 \circ x = c$.

Example 10. Let $d = 0.2$ and

$$a^3 = \left[\begin{array}{ccc} 0.2 & 0.8 & 0.2 \end{array} \right].$$

We solve equation $a^3 \circ x = d$. It means

$$(0.2 \wedge x_1) \vee (0.8 \wedge x_2) \vee (0.2 \wedge x_3) = 0.2.$$

We have
• $0.2 \wedge x_1 \leqslant 0.2$ for $x_1 \in [0, 1]$ and $0.2 \wedge x_1 = 0.2$ for $x_1 \in [0.2, 1]$,
• $0.8 \wedge x_2 \leqslant 0.2$ for $x_2 \in [0, 0.2]$, $0.8 \wedge x_2 = 0.2$ for $x_2 = 0.2$ and $0.8 \wedge x_2 > 0.2$ for $x_2 \in (0.2, 1]$,
• $0.2 \wedge x_3 \leqslant 0.2$ for $x_3 \in [0, 1]$, $0.2 \wedge x_3 = 0.2$ for $x_3 \in [0.2, 1]$.
If $x_1 \in [0.2, 1]$, then $x_2 \in [0, 0.2]$, $x_3 \in [0, 1]$ are arbitrary. It means

$$x \in [0.2, 1] \times [0, 0.2] \times [0, 1].$$

If $x_2 = 0.2$, then $x_1, x_3 \in [0, 1]$ are arbitrary, it means

$$x \in [0, 1] \times \{0.2\} \times [0, 1].$$

If $x_3 \in [0.2, 1]$, then $x_1 \in [0, 1]$, $x_2 \in [0, 0.2]$ are arbitrary, it means

$$x \in [0, 1] \times [0, 0.2] \times [0.2, 1],$$

Putting

$$v^3 = \begin{bmatrix} 0.2 \\ 0 \\ 0 \end{bmatrix}, \quad w^3 = \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix}, \quad y^3 = \begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix}, \quad u^3 = \begin{bmatrix} 1 \\ 0.2 \\ 1 \end{bmatrix}$$

we have solution set in the form

$$S_3 = [v^3, u^3] \cup [w^3, u^3] \cup [y^3, u^3].$$

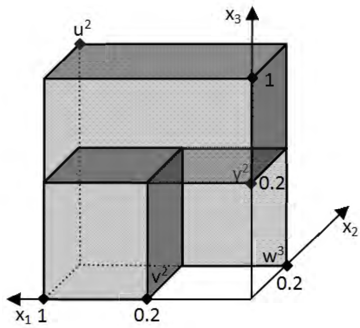

Figure 3: Set of solutions $a^3 \circ x = d$.

Example 11. Let $e = 0.8$ and

$$a^1 = [0.5 \quad 0.3 \quad 0.7].$$

Set of solution $a^1 \circ x = e$ is the empty set, because $a^1 \circ x \leqslant a^1 \circ \mathbf{1} = 0.7 < 0.8 = e$ for arbitrary $x \in [0, 1]^3$ (based on (9)).

Because we want to consider LLR and FSE, we join results from Examples 8-10.

Example 12. Consider the following fuzzy linear equations

$$a^1 \circ x = c, \quad a^2 \circ x = c \tag{29}$$

based on examples 8 and 9. Solution of such system of equations is intersection of solution sets of FLE calculated in Examples 8 and 9. It means

$$S_5 = S_1 \cap S_2 = ([v^1, u^1] \cup [w^1, u^1]) \cap ([v^2, u^2] \cup [w^2, u^2] \cup [y^2, u^2]) =$$

$([v^1,u^1]\cap[v^2,u^2])\cup([v^1,u^1]\cap[w^2,u^2])\cup([v^1,u^1]\cap[y^2,u^2])\cup([w^1,u^1]\cap[v^2,u^2])$ $\cup([w^1,u^1]\cap[w^2,u^2])\cup([w^1,u^1]\cap[y^2,u^2]) = [v^1\vee v^2, u^1\wedge u^2]\cup[v^1\vee w^2, u^1\wedge u^2]$ $\cup[v^1\vee y^2, u^1\wedge u^2]\cup[w^1\vee v^2, u^1\wedge u^2]\cup[w^1\vee w^2, u^1\wedge u^2]\cup[w^1\vee y^2, u^1\wedge u^2]$, where $u^1\wedge u^2 = u^2$, $v^1\vee y^2 = y^2$, $w^1\vee v^2 = v^2$, $w^1\vee y^2 = v^1\vee v^2$ and

$$v^1\vee v^2 = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}, \quad v^1\vee w^2 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}, \quad w^1\vee w^2 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}.$$

From this we obtain

$$\begin{aligned} S_5 = [v^1\vee v^2, u^2]\cup[v^1\vee w^2, u^2]\cup[y^2, u^2]\cup[v^2, u^2]\cup[w^1\vee w^2, u^2] \\ = [y^2, u^2]\cup[v^2, u^2]. \end{aligned} \tag{30}$$

Let us note that system of equations (29) is equivalent to the following FSE $A\circ x = b$, where

$$A = \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 & 0.7 \\ 0.5 & 0.6 & 0.8 \end{bmatrix}, \quad b = \begin{bmatrix} c \\ c \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

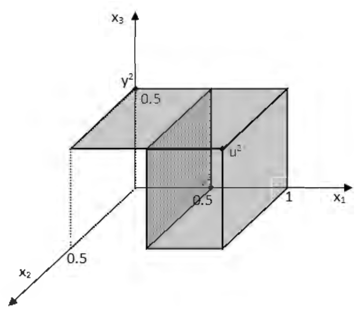

Figure 4: Solution set of $A\circ x = b$ from example 12.

Example 13. Consider the following fuzzy linear equations

$$a^2\circ x = c, \quad a^3\circ x = d \tag{31}$$

based on Examples 9 and 10. Solution of such system of equations is intersection of solution sets of FLE calculated in Examples 9 and 10. It means

$S_6 = S_2\cap S_3 = ([v^2,u^2]\cup[w^2,u^2]\cup[y^2,u^2])\cap([v^3,u^3]\cup[w^3,u^3]\cup[y^3,u^3]) =$

$$([v^2,u^2]\cap[v^3,u^3])\cup([v^2,u^2]\cap[w^3,u^3])\cup([v^2,u^2]\cap[y^3,u^3])\cup([w^2,u^2]\cap[v^3,u^3])$$
$$\cup([w^2,u^2]\cap[w^3,u^3])\cup([w^2,u^2]\cap[y^3,u^3])\cup([y^2,u^2]\cap[v^3,u^3])$$
$$\cup([y^2,u^2]\cap[w^3,u^3])\cup([y^2,u^2]\cap[y^3,u^3]) = [v^2\vee v^3,u^2\wedge u^3]\cup[v^2\vee w^3,u^2\wedge u^3]$$
$$\cup[v^2\vee y^3,u^2\wedge u^3]\cup[w^2\vee v^3,u^2\wedge u^3]\cup[w^2\vee w^3,u^2\wedge u^3]\cup[w^2\vee y^3,u^2\wedge u^3]$$
$$\cup[y^2\vee v^3,u^2\wedge u^3]\cup[y^2\vee w^3,u^2\wedge u^3]\cup[y^2\vee y^3,u^2\wedge u^3],$$
where $v^2 \vee v^3 = v^2$, $w^2 \vee w^3 = w^2$, $y^2 \vee y^3 = y^2$ and

$$u^2\wedge u^3 = \begin{bmatrix} 1 \\ 0.2 \\ 0.5 \end{bmatrix}, v^2\vee w^3 = \begin{bmatrix} 0.5 \\ 0.2 \\ 0 \end{bmatrix}, v^2\vee y^3 = \begin{bmatrix} 0.5 \\ 0 \\ 0.2 \end{bmatrix},$$

$$w^2\vee v^3 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0 \end{bmatrix}, w^2\vee y^3 = \begin{bmatrix} 0 \\ 0.5 \\ 0.2 \end{bmatrix}, y^2\vee v^3 = \begin{bmatrix} 0.2 \\ 0 \\ 0.5 \end{bmatrix},$$

$$y^2\vee w^3 = \begin{bmatrix} 0 \\ 0.2 \\ 0.5 \end{bmatrix}.$$

Since we have $w^2||(u^2 \wedge u^3)$, $(w^2 \vee v^3)||(u^2 \wedge u^3)$, therefore we get

$$\begin{aligned} S_6 &= [v^2,u^2\wedge u^3]\cup[v^2\vee w^3,u^2\wedge u^3]\cup[v^2\vee y^3,u^2\wedge u^3]\cup[y^2,u^2\wedge u^3] \\ &= [v^2,u^2\wedge u^3]\cup[y^2,u^2\wedge u^3]. \end{aligned} \tag{32}$$

S_6 is solution of system of equations (31). Note that system of equations in the form (31) is equivalent to the following FSE: $A \circ x = b$, where

$$A = \begin{bmatrix} a^2 \\ a^3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.6 & 0.8 \\ 0.2 & 0.8 & 0.2 \end{bmatrix}, \quad b = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}.$$

Example 14. Consider the following fuzzy linear equations

$$a^1\circ x = c, \quad a^2\circ x = c, \quad a^3\circ x = d \tag{33}$$

based on Examples 8, 9, 10. Solution of such system of equations is intersection of solution sets of FLE calculated in Examples 8, 9, 10. Using the results from (30) and (32) and results from Examples 12 and 13 we have

$$S_7 = S_1\cap S_2\cap S_3 = ([y^2,u^2]\cup[v^2,u^2])\cap([v^3,u^2\wedge u^3]\cup[y^2,u^2\wedge u^3]) =$$

$([y^2, u^2] \cap [v^3, u^2 \wedge u^3]) \cup ([v^2, u^2] \cap [v^3, u^2 \cap u^3]) \cup ([y^2, u^2] \cap [y^2, u^2 \wedge u^3])$
$\cup([y^2, u^2] \cap [v^3, u^2 \cap u^3]) = [y^2 \vee v^3, u^2 \wedge u^3] \cup [v^2 \vee v^3, u^2 \wedge u^3] \cup [y^2, u^2 \wedge u^3]$,
where $v^2 \vee v^3 = v^2$ and $u^2 \wedge u^3 = u^3$, $v^2 \vee y^3$ as in the example 12. Because $[y^2 \vee v^3, u^2 \wedge u^3] \subset [v^2, u^2 \wedge u^3]$, therefore solution of system (33) is

$$S_7 = [v^2, u^2 \wedge u^3] \cup [y^2, u^2 \wedge u^3]. \tag{34}$$

This is equivalent to FSE in the form $A \circ x = b$, where

$$A = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 & 0.7 \\ 0.5 & 0.6 & 0.8 \\ 0.2 & 0.8 & 0.2 \end{bmatrix}, \quad b = \begin{bmatrix} c \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.2 \end{bmatrix}.$$

From this examples we have

Corollary 7. *Let $A \in [0,1]^{m \times n}$, $b \in [0,1]^m$, $* = \wedge$. Solution family of considered fuzzy systems of equations $A \circ x = b$ is an intersection of solution sets of suitable fuzzy linear equations. It means that solution set of $A \circ x = b$ has the form:*

$$S = \bigcap_{i=1}^{m} (\bigcup_{k=1}^{n_i} [v^{i,k}, u^i]) \tag{35}$$

for the proper parameters n_i and the vectors $v^{i,k} \leqslant u^i$.

Some properties of solution FSE with $\max - \min$ product of this form was described in [5].

Since nonempty intersections of intervals in (35) gives intervals, then based on examples 12-14, we have

Corollary 8. *Solution family of considered fuzzy systems of equations (and similarly for fuzzy linear equations) is a sum of intervals. Moreover, the set of all solutions in the form (35) has the greatest solution and can have different minimal solutions.*

4 Concluding remark

The above Corollaries 7, 8 will be treated as conjectures in further investigation of fuzzy system of equations with $\max - *$ product, where the operation $*$ is continuous and increasing operations.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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