Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Recent Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
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Central limit theorem on IV-events

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Abstract

IVF-events is important notion. Interval valued event (IVF event) is a pair $A = (\mu_A, \nu_A)$ of fuzzy events such that $\mu_A \leq \nu_A$. The paper contains one main result: central limit theorem. At the begin we define joint observable.

Keywords: IV-events, central limit theorem.

1 Introduction

We shall start with a measurable space (Ω, S), where Ω is a non-empty set and S a σ-algebra of subsets of Ω, i.e. S is closed under complements and countable unions. Usually a fuzzy event is a measurable mapping $f : \Omega \to [0,1]$, i.e. $f^{-1}(J) = \{\omega \in \Omega; f(\omega) \in J\}$ is a set belonging to S for every interval $J \subset [0,1]$. Interval valued event (IVF event) is a pair $A = (\mu_A, \nu_A)$ of fuzzy events (i.e. $\mu_A, \nu_A : (\Omega, S) \to [0,1]$ are fuzzy events such that $\mu_A \leq \nu_A$. Let F be the set of all IVF events.

Definition 1 *We define two binary operations* $\boxplus, \boxdot$*:* $F \times F \longrightarrow F$ *as follows:*

$$A \boxplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B) \wedge 1),$$
$$A \boxdot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B - 1) \vee 0),$$

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and a partial ordering as follows

$$A \le B \Longleftrightarrow \mu_A \le \mu_B, \nu_A \le \nu_B.$$

Evidently $(0_\Omega, 0_\Omega)$ *is the least element of* F, $(1_\Omega, 1_\Omega)$ *is the greatest element of* F.

In the classical probability space (Ω, S, P) a random variable is consider as an S-measurable mapping

$$\xi : \Omega \longrightarrow R,$$

if $I \subset R$ is an interval then $\xi^{-1}(I) \in S$.

Definition 2 *An observable is a mapping*

$$x : \sigma(J) \longrightarrow F$$

satisfying the following conditions
i) $x(R) = (1,1), x(\emptyset) = (0,0)$,
ii) $A \cap B = \emptyset \Rightarrow x(A) \boxdot x(B) = (0,0)$, $x(A \cup B) = x(A) \boxplus x(B)$
iii) $A_n \nearrow A \Rightarrow x(A_n) \nearrow x(A)$

Definition 3 *Every mapping* $m : F \longrightarrow [0,1]$ *satisfying the conditions:*
i) $m(0_\Omega, 0_\Omega) = (0), m(1_\Omega, 1_\Omega) = (1)$
ii) $A \boxdot B = (0_\Omega, 0_\Omega) \Longrightarrow m(A \boxplus B) = m(A) + m(B)$
iii) $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$.

Proposition 1 *If* $x : \sigma(J) \longrightarrow F$ *is an observable, and* $m : F \to \langle 0,1 \rangle$ *is a state, then*

$$m_x = m \circ x : \sigma(J) \to \langle 0,1 \rangle,$$

defined by

$$m_x(A) = m(x(A))$$

is a probability measure.

Proof 1
i) $m_x(R) = m(x(R)) = m(1,1) = 1$
ii) If $A \cap B = \emptyset$, *then* $x(A) \boxdot x(B) = (0,0)$ *hence*

$$m_x(A \cup B) = m(x(A \cup B)) = m(x(A) \boxplus x(B)) = m(x(A)) + m(x(B)) = m_x(A) + m_x(B).$$

iii) $A_n \nearrow A$ *implies* $x(A_n) \nearrow x(A)$ *hence*

$$m_x(A_n) = m(x(A_n)) \nearrow m(x(A)) = m_x(A).$$

Proposition 2 *Let* $x : \sigma(J) \longrightarrow F$ *be an observable,* $m : F \longrightarrow [0,1]$ *be a state. Define* $F : R \longrightarrow [0,1]$ *by the formula*

$$F(s) = m(x(-\infty, s)).$$

Then F *is non-decreasing, left continuous in any point* $s \in R$,

$$lim_{s \to \infty} F(s) = 1 \text{ , } lim_{s \to -\infty} F(s) = 0.$$

Proof 2 *If* $s < t$, *then* $x((-\infty, t)) = x((-\infty, s)) \boxplus x(\langle s, t\rangle) \geq x((-\infty, s))$ *hence*

$$F(t) = m((-\infty, t)) \geq m(x((-\infty, s)) = F(s),$$

F *is non decreasing. If* $s_n \nearrow s$ *then* $x((-\infty, s_n)) \nearrow x((-\infty, s))$, *hence*

$$F(s_n) = m(x((-\infty, s_n))) \nearrow m(x((-\infty, s))) = F(s),$$

F *is left continuous in any* $s \in R$.

Similarly,

$$s_n \nearrow \infty \Longrightarrow x((-\infty, s_n)) \nearrow x((-\infty, \infty)) = (1, 1).$$

Therefore

$$F(s_n) = m(x((-\infty, s_n))) \nearrow s_n((1, 1)) = 1$$

for every $s_n \nearrow \infty$, *hence* $lim_{s \to \infty} F(s) = 1$.

Similarly, we obtain $s_n \searrow -\infty \Longrightarrow -s_n \nearrow \infty$, *hence*

$$m(x((s_n, -s_n))) \nearrow m(x(R)) = 1.$$

$$1 = lim_{n \to \infty} F(-s_n) = lim_{n \to \infty}(x(\langle s_n, -s_n))) + lim_{n \to \infty} F(s_n) = 1 + lim_{n \to \infty} F(s_n),$$

hence

$$lim_{n \to \infty} F(s_n) = 0$$

for any $s_n \searrow -\infty$.

2 Central limit theorem

If we want to define the sum $\xi + \eta$ of two observables, one of possibilities is the following formulation. Put

$$T = (\xi, \eta) : \Omega \longrightarrow R^2,$$
$$g : R^2, g(s,t) = s + t,$$
$$\xi + \eta = g \circ T : \Omega \longrightarrow \Omega.$$

Namely, it is convenient for the constructing of preimages:

$$(\xi + \eta)^{-1}(A) = T^{-1}(g^{-1}(A)).$$

In our IV-case, we have two observables

$$x, y : \sigma(J) \longrightarrow F,$$

hence $x + y$ could be define as a morphism

$$(x + y)(A) = h(g^{-1}(A)),$$

where $h : \sigma(J_2) \longrightarrow F$ is a morphism connecting with x, y. In the classical case it was realiyed by the formula

$$T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D).$$

In our IV-case, instead of intersection, we shall use the product of IV-sets.

$$A \boxdot B = (\mu_A, \nu_A) \boxdot (\mu_B, \nu_B) = (\mu_A.\mu_B, (1 - \nu_A).(1 - \nu_B)) =$$
$$(\mu_A.\mu_B, 1 - \nu_A + \nu_B - \nu_A.\nu_B).$$

Definition 4 *Let $x_1, x_2, ..., x_n : \sigma(J) \longrightarrow F$ be observables. By the joint observables of $x_1, x_2, ..., x_n$ we consider a mapping $h : \sigma(J^n) \longrightarrow F$ (J^n being the set all intervals of R^n satisfying the following conditions:*

i) $h(R^n) = (0, 1)$

ii) $A \cap B = \emptyset \Longrightarrow h(A \cup B) = h(A) \boxplus h(B)$, and $h(A) \boxdot h(B) = (0, 1)$

iii) $A_n \nearrow A \Longrightarrow h(A_n) \nearrow h(A)$

iv) $h(C_1 \times C_2 \times ...C_n) = x_1(C_1).x_2(C_2.....x_n(C_n))$, for any $C_1, C_2, ...C_n \in J$.

Theorem 1 *For any observables $x_1, x_2, ..., x_n : \sigma(J) \longrightarrow F$ there exits their joint observable $h : \sigma(J^n) \longrightarrow F$.*

Proof 3 *We shall prove it for $n = 2$. Consider two observables $x, y : \sigma(J) \longrightarrow F$. Since $x(A) \in F$, we shall write*

$$x(A) = (x^\flat(A), x^*(A))$$

and similarly

$$y(B) = (y^\flat(B), y^*(B)).$$

By the definition of product $x(C).y(D)$, we have

$$x(C).y(D) = (x^\flat(C), x^*(C)).(y^\flat(D), y^*(D)) =$$
$$= (x^\flat(C).y^\flat(D), (1 - (1 - x^*(C))).(1 - (1 - x^*(D)))) =$$
$$= (x^\flat(C).y^\flat(D), x^*(C).y^*(D)).$$

Therefore we shall construct similarly

$$(h^\flat(K), h^*(K).$$

Fix $\omega \in \Omega$ and put

$$\mu_A = x^\flat(A)(\omega),$$
$$\nu_B = x^\flat(B)(\omega),$$
$$h^\flat(K) = \mu \times \nu(K).$$

$\mu \times \nu$ is the product of probability measures μ, ν. Then

$$h^\flat(C \times D)(\omega) = \mu \times \nu(C \times D) = \mu(C).\nu(D) = x^\flat(C).y^\flat(D)(\omega),$$

hence

$$h^\flat(C \times D) = x^\flat(C).y^\flat(D).$$

Analogously

$$h^*(C \times D) = x^*(C).y^*(D).$$

If we define

$$h(A) = (h^\flat(A), h^*(A)), A \in \sigma(J^2),$$

then

$$h(C \times D) = (x^\flat(C), y^\flat(D), x^*(C).y^*(D)) = x(C).y(D).$$

Then previous theorem can be applied for obtaining the sum

$$x_1 + x_2 + ... + x_n = h \circ g^{-1}$$

with

$$g(u_1, ...u_n) = u_1 + ... + u_n$$

or for the arithmetic means

$$\frac{1}{n}(x_1 + x_2 + ... + x_n) = h \circ g^{-1},$$

with

$$g(u_1, ...u_n) = \frac{1}{n}(u_1 + ... + u_n)$$

Consider again a probability measure space (Ω, S, P) and a sequence $(\xi_n)_n$ of square integrable, equally distributed variables with $E(\xi_n) = a$, $D(\xi_n) = \sigma^2$ $n = 1, 2, ...$ Then

$$lim_{n\to\infty}P(\omega \in \Omega; \frac{\frac{1}{n}\sum_{i=1}^{n}\xi_i(\omega) - a}{\frac{\sigma}{\sqrt{n}}} < t) = \phi(t)$$

for any $t \in R$. (Here $\phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{\frac{-u^2}{2}}du$.) We shall translate the theorem in our IV-case.

Definition 5 *Let $m : F \longrightarrow \langle 0, 1\rangle$ be a state, $(x_n)_{n=1}^{\infty}$ be a sequence of observables, $h_n : \sigma(J_n) \to F$ be the joint observable of $x_1, x_2, ..., x_n, (n = 1, 2, ...)$. Then $(x_n)_n$ is called independent, if*

$$m(h_n(C_1 \times C_2 \times ... \times C_n)) = m(x_1(C_1)).m(x_2(C_2))...m(x_n(C_n))$$

for any $n \in N$ and any $C_1, ..., C_n \in \sigma(J)$.

Theorem 2 *Let $(x_n)_{n=1}^{\infty}$ be a sequence of square integrable, equally distributed, independent observables, with $E(x_n) = a$, $D(x_n) = \sigma^2$ $(n = 1, 2, ...)$. Then*

$$lim_{n\to\infty} = (\frac{\sqrt{n}}{\sigma})(\frac{1}{n}\sum_{i=1}^{n} x_i - a)((-\infty, t)) = \phi(t)$$

for any $t \in \Omega$.

Proof 4 *Put $g_n : R^n \to R$*

$$g_n(u_1, u_2, ..., u_n) = \frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum_{i=1}^{n} u_i - a)$$

so that

$$\frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum_{i=1}^{n} u_i - a)(-\infty, t) = g_n^{-1}((-\infty, t))$$

and

$$\frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum_{i=1}^{n} u_i - a)(-\infty, t) = h_n \circ g_n^{-1}.$$

Consider now sequence $(m \circ h_n)_n$ of probability measures $m \circ h_n : \sigma(J_n) \to \langle 0, 1\rangle$. By the definition of h_n we have

$$(m \circ h_{n+1})(A \times R) = (m \circ h_n)(A), A \in \sigma(J^n).$$

Therefore $(m \circ h_n)_n$ forms a consisting system of probability measures

$$m \circ h_n : \sigma(J^n) \to \langle 0, 1\rangle.$$

Consider the space R^N, the projections $\Pi_n : R^N \to R^n$

$$\Pi_n((u_i)_{i=1}^{\infty}) = (u_1, ...u_n)$$

and the family of all cylinders in R^N, i.e. sets of the form

$$\varepsilon = \{\Pi_n^{-1}; n \in N, A \in \sigma(J^n)\}.$$

By the Kolmogorov consistency theorem there exists a probability measure $P : \sigma(\varepsilon) \to \langle 0, 1\rangle$ such that $P \circ \Pi_n^{-1} = m \circ h_n$ for any $n \in N$. Return now to our sequence $(x_n)_{n=1}^{\infty}$ of observables. Define on R^N the sequence $(\xi_n)_{n=1}^{\infty}$ by the formula

$$\xi_n((u_{i=1}^{\infty}) = u_n$$

Then

$$m(x_n(C)) = m(h_n(R \times ... \times R \times C \times R \times ... \times R)) =$$
$$P(\Pi_n^{-1}(R \times ... \times R \times C \times R \times ... \times R)) = P(\xi_n^{-1}(C)).$$

Therefore

$$E(\xi_n) = \int_{-\infty}^{\infty} tdm_{\xi_n}(t) = \int_{-\infty}^{\infty} tdm_{\xi_n}(t) = E(x_n)$$

and similarly

$$D(\xi_n) = D(x_n).$$

Moreover

$$P(\xi_1^{-1}(C_1) \cap \ldots \cap \xi_n^{-1}(C_n)) = P(\Pi_n^{-1}(C_1 \times \ldots \times C_n)) =$$
$$= m(h_n(C_1 \times \ldots \times C_n)) = m(x_1(C_1)) \ldots m(x_n(C_n)) =$$
$$= P(\xi_1^{-1}(C_1)) \ldots P(\xi_n^{-1}(C_n))$$

hence $\xi_1, \xi_2, \ldots$ *are independent. Put* $g_n : R^n \to R$ *by the formula*

$$g_n(t_1, t_2, ..., t_n) = \frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum_{i=1}^{n} u_i - a),$$

$$\eta_n = g_n(\xi_1, ..., \xi_n) = \frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum_{i=1}^{n} u_i - a) = g_n \circ \Pi_n.$$

Then

$$lim_{n\to\infty} P(\eta_n^{-1}((-\infty, t))) = \phi(t)$$

for any $t \in R$*. But*

$$\frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum_{i=1}^{n} u_i - a) = h_n \circ g_n^{-1}.$$

Therefore

$$lim_{n\to\infty} m(\frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum_{i=1}^{n} u_i - a))(-\infty, t) =$$
$$lim_{n\to\infty} m(h_n(g_n^{-1}(-\infty, t))) =$$
$$lim_{n\to\infty} P(\Pi_n^{-1}(g_n^{-1})((-\infty, t)))) =$$
$$lim_{n\to\infty} P(\eta_n^{-1}((-\infty, t))) = \phi(t).$$

References

[1] Riečan, B. - Kráľ, P.: Probability on interval valued events.

[2] Riečan, B. - Petrovičová, J.: On the Lukasiewicy probability theory on IF-sets. Tatra Mt. Math. Publ. 46(2010), 125-146

[3] Ciungu, L. - Riečan, B.: Representation theorems for probabilities in IFS-events. Inform. Sci. 180, 2010, 793-798.

[4] Dvurečenskij, A - Pulmannová, S.: Trends in Quantum Structures. Kluwer Academic Press, Dordrecht, 2000.

[5] Lendelová, K.: Central limit theorem for L-posets. Journal of Electrical engineering 56(2005), 7-9.

[6] Riečan, B.: Probability theory on IF events. In: Algebraic and Proof-Theoretic Aspects of Non-classical. Logics (S.Aguzzolii, et.al.,eds.). Papers in honor of Daniele Mundici on the occasion of his 60th birthday. Lecture Notes in Comput. Sci., Springer, Berlin, 2007, pp. 290-308.

[7] Riečan, B.-Mundici, D.: Probability on MV-algebras. In: Handbook on Measure Theory, Elsevier, Amsterdam, 2002, pp. 869-909.

[8] Valenčaková, V.: A note on the conditional probability of IF events. Math. Slovaca 59 (2009), 251-260.

[9] Vrábelová, M.: On the conditional probability in product MV-algebras. Soft Computing 4 (2000), 58-61.

The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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