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Abstract We indicate some possible tools and techniques of fuzzy sets theory which can be of use in extensions of some more popular optimization and mathematical programming. We conWe briefly survey main issues and developments in fuzzy optimization to make iot possible to deal with imprecise data. We mainly concern linear programming due to its great practical relevance.

Key words: fuzzy optimization, fuzzy linear programming, fuzzy integer programming, fuzzy 0–1 programming, fuzzy dynamic programming, fuzzy multiobjective mathematical programming.

1 Introduction

Optimization belongs to a much wider class of *decision making* problems whose essence may be summarized as follows:

- there is a set of feasible options (alternatives, variants , ...),
- there is some mechanism for the representation of preferences among the options which is given, e.g., by pairwise comparisons, preference orderings, utility functions, etc.,
- there is some choice (rationality) criterion determining which options should be chosen (e.g. those with the highest value of a utility functions).

In *optimization* problems information on the preferences among options is described by a utility (performance, objective, ...) function which maps a given set of feasible options into the real line, hence the comparison is straightforward and natural, i.e. the greater (or lower) the value of this function the better.

The set of feasible options in an optimization problem is often described by a system on equations and/or inequalities, and in such a case the problem is referred to as one of *mathematical programming*.

Methods and techniques of optimization, or - more specifically - those of mathematical programming have been successfully employed for years in various areas, mostly in problems with relatively well-defined structures and data, the so-called *hard* problems. This has allowed the formulation of optimization problems with precisely specified constraints and objective functions which are solvable by relatively efficient traditional analytical and computational tools and techniques.

Unfortunately, attempts to employ optimization tools for solving the so-called *soft* problems in which goals and constraints are not clear-cut, a key role is played by human value systems and judgments, there are multiple decision makers and criteria, dynamics is involved, etc. have not been so successful.

It can be argued that one of major obstacles in the application of traditional optimization tools in soft problems is the predominatly subjective nature of available information, which is due to the criticality of human factors, and its imprecise form which is due to the use of natural language that is the only fully natural human means of communication.

Developments of fuzzy sets theory have given more and more evidence that this theory may provide useful means for a more adequate, effective and efficient handling of optimization problems in soft environments. It is in this sense that fuzzy sets theory will be advocated as a promising tool for *softening* traditional optimization models and techniques.

In this paper we will present a brief account of the state of the art of fuzzy optimization and mathematical programming. We will present major concepts, ideas and developments, and refer the reader to more relevant literature. We should bear in mind that the literature in this field is voluminous, and cannot be cited in full in this short review. However, luckily enough there exist in the literature two edited volumes devoted to fuzzy optimization, the former one [, (Kacprzyk and Orlovski 1987a)], and the recent one [, (Delgado, Kacprzyk, Verdegay and Vila 1994)] which are sources of information on virtually all newer and more relevant developments. Other books deal with more specific areas as, e.g., [, (Fedrizzi, Kacprzyk and Roubens 1991)] is a source on interactive approaches, and the recent book [, (Kacprzyk 1997)] is the source on fuzzy dynamic programming.

What concerns other more extensive surveys on fuzzy optimization and fuzzy mathematical programming, the interested reader is referred to, e.g., [, (Kacprzyk and Orlovski 1987b)] or [, (Fedrizzi, Kacprzyk and Verdegay 1991)] or [, Delgado, Verdegay and Vila 1994].

We will start with a discussion on some preliminaries which constitute a point of departure for fuzzy optimization models, mainly of the [, (Bellman and Zadeh 1970)] approach. Then, we present in more detail fuzzy linear programming due to its relevance. We sketch fuzzy nonlinear programming, including fuzzy integer, 0–1, geometric, fractional, etc. programming. Then we outline fuzzy dynamic programming. Finally, we consider some approaches to fuzzy multiobjective mathematical programming.

2 Approaches to fuzzy optimization

In this section we will discuss some general issues related to fuzzy optimization. Basically, we will assume that both the feasible set of options and the evaluation of options will be fuzzily described. This will provide a point of departure to fuzzy mathematical programming, with the feasible set of options specified by a set of equalities and/or inequalities.

2.1 Approaches to fuzzy optimization with an explicitly specified feasible set

As we have already mentioned in Section 1, in an optimization problem the following two elements are crucial:

- a set of feasible options (alternatives, variants, ...) also termed a feasible set, and
- an objective (performance) function employed for the comparison and ranking of the alternatives in order to find an optimal one(s).

In this section we will briefly analyze these elements in a more general context of fuzzy optimization. As we will see in Section 3, in which we will discuss issues related to fuzzy mathematical programming, these elements will also occur there.

The class of fuzzy optimization problems considered here may be stated as follows. Suppose that $X = \{x\}$ is a set of options. The objective function is $F : X \longrightarrow L(R)$ where $L(R)$ is a family of fuzzy sets defined in R, the real line; i.e. $F(x)$ is a fuzzy number which provides an imprecise (fuzzy) evaluation of option $x \in X$. The set of feasible options is imprecisely specified by a fuzzy set C in X such that $C(x) \in [0,1]$ stands for the degree of feasibility, from 1 for fully feasible to 0 for fully infeasible, through all intermediate values.

The optimization problem may then be generally denoted as

$$\widetilde{\max}_{x \widetilde{\in} C} F(x) \tag{1}$$

which is to be read as: find a *possibly high* ($\widetilde{\max}$) value of F over the x's "belonging" ($\widetilde{\in}$) to the (fuzzy) feasible set C.

The above general problem formulation may be formally stated in various ways exemplified by one in terms of [, (Bellman and Zadeh 1970)] general approach to decision making under fuzziness, and one in terms of a representation of the fuzzy feasible set via α-cuts.

2.2 Attainment of fuzzy goals and satisfaction of fuzzy constraints – Bellman and Zadeh's approach

A general approach to decision making under fuzziness (or, as originally termed, in a fuzzy environment) [, (Bellman and Zadeh 1970)] is a powerful framework which is a point of departure for an overwhelming majority of fuzzy decision making, optimization, control, etc. models. It is also a convenient apparatus for the class of fuzzy optimization problems (1) considered here. This approach was discussed in Section F5.1, and we will present here again its main elements tailored to our specific needs.

In this approach we have an explicitly specified fuzzy feasible set, called a *fuzzy constraint*, and an explicitly specified fuzzy set of options which attain the goal, called a *fuzzy goal*. The fuzzy constraint is characterized by its membership function $C(x)$ such that $C(x) = 1$ stands for a fully feasible x, $C(x) = 0$ stands for a fully infeasible x, and intermediate values denote feasibility to a degree, from 0 to 1. And, similarly, the fuzzy goal is characterized by its membership function $G(x)$, with an analogous interpretation.

The fuzzy goal $G(x)$ is usually assumed to be of the following type

$$G(x) = \begin{cases} 1 & \text{for } f(x) \geq \overline{f} \\ g(x) & \text{for } \underline{f} < f(x) < \overline{f} \\ 0 & \text{for } f(x) \leq \underline{f} \end{cases} \tag{2}$$

which is to be read as: we are fully satisfied [$G(x) = 1$] with the x's for which the objective (performance) function $f(x)$ attains a value at least equal to a satisfaction level $\overline{f}$, we are partially satified (to degree $0 < G(x) = g(x) < 1$) with the x's for which $f(x)$ is between the satisfaction level $\overline{f}$ and the lowest admissible level $\underline{f}$, and we are fully dissatified [$G(x) = 0$] with the x's for which $f(x)$ is below the lowest admissible level $\underline{f}$.

Notice that the above definition in terms of satisfaction levels is intuitively appealing, and has proven to be extremely useful in applications [cf. [, (Kacprzyk 1997)]].

The problem is now stated as to

"satisfy the fuzzy constraint **and** attain the fuzzy goal"

which, by introducing the concept of a *fuzzy decision*, D, may be written as

$$D(x) = C(x) \wedge G(x) = \min[C(x), G(x)] \tag{3}$$

with the understanding that "$\wedge$, i.e. the minimum, which reflects the traditional definition of the intersection of fuzzy sets, may readily be replaced by another operation as, say, a t-norm.

The fuzzy decision $D(x)$ given by (3) specifies therefore a fuzzy set of options which satisfy the fuzzy constraint and attain the fuzzy goal. Normally, since even if the problem is fuzzy its solution to be implemented must be crisp, we wish to find a nonfuzzy option (or options) which best satifies the fuzzy constraint and attain the fuzzy goal, and this gives rise to the concept of an optimal (maximizing) decision, $x^* \in X$, to be determined

$$D(x^*) = \sup_{x \in X} D(x) = \sup_{x \in X} [C(x) \wedge G(x)] \tag{4}$$

One may readily notice that an analogous line of reasoning can be applied for multiple fuzzy constraints, $C^1(x), \ldots, C^m(x)$, and multiple fuzzy goals, $G^1(x), \ldots, G^n(x)$, and we obtain then

$$D(x) = C^1(x) \wedge \ldots \wedge C^m(x) \wedge G^1(x) \wedge \ldots \wedge G^n(x) \tag{5}$$

while an optimal (maximizing) decision to be found, $x^* \in X$, is given by

$$\begin{aligned} D(x^*) &= \\ &= \sup_{x \in X} [C^1(x) \wedge \ldots \wedge C^m(x) \wedge G^1(x) \wedge \ldots \wedge G^n(x)] \end{aligned} \tag{6}$$

Moreover, if the fuzzy constraint is defined in $X = \{x\}$, $C(x)$, and the fuzzy goal is defined in $Y = \{y\}$, $G(y)$, and there is a function $h : X \longrightarrow Y$, $y = h(x)$, then we denote by $\overline{G}[h(x)]$ the fuzzy goal in X induced by the fuzzy goal $G(y)$ in Y, then then fuzzy decision (3) becomes

$$D(x) = C(x) \wedge \overline{G}[h(x)] \tag{7}$$

and we seek an optimal (maximizing) decision $x^* \in X$ such that

$$D(x^*) = \sup_{x \in X} D(x) = \sup_{x \in X}(C(x) \wedge \overline{G}[h(x)]) \tag{8}$$

And analogously, for multiple fuzzy constraints and multiple fuzzy goals defined in X and Y, respectively, we obtain the fuzzy decision

$$D(x) = C^1(x) \wedge \ldots \wedge C^m(x) \wedge \overline{G}^1[h(x)] \wedge \ldots \wedge \overline{G}^n[h(x)] \tag{9}$$

and an optimal (maximizing) decision to be found, $x^* \in X$, is given by

$$\begin{aligned} D(x^*) = \max_{x \in X} D(x) = \max_{x \in X}(C^1(x) \wedge \ldots \\ \ldots \wedge C^m(x) \wedge \overline{G}^1[h(x)] \wedge \ldots \wedge \overline{G}^n[h(x)]) \end{aligned} \tag{10}$$

Notice that in the above general Bellman and Zadeh's model the values of the objective function, $f(x)$, are nonfuzzy [cf. (2)], and only its maximization is imprecisely specified. In an approach proposed in [, (Orlovski 1980)] it is possible to extend the above general model to the case when the values of the objective function are fuzzy, characterized by membership functions $g : X \times R \longrightarrow [0,1]$ such that for each value of $x \in X$ the objective function may take on different real values, with different degrees of membership from $[0,1]$.

Basically, in [, (Orlovski 1980)] the following sets are introduced:

$$N = \{(x,r) : (x,r) \in X \times R, g(x,r) > G(x)\} \tag{11}$$

$$N_x = \{r : r \in R, (x,r) \in N\} \tag{12}$$

$$X^0 = \{x : x \in X, N_x \neq \emptyset\} \tag{13}$$

the fuzzy decision is defined as

$$D(x) = \begin{cases} C(x) \wedge \inf_{r \in N} G(x) & \text{for } x \in X^0 \\ C(x) & \text{otherwise} \end{cases} \tag{14}$$

and an optimal (maximizing) decision $x^* \in X$ is sought such that

$$D(x^*) = \max_{x \in X} D(x) \tag{15}$$

2.3 Using the α-cuts of the fuzzy feasible set

A common approach in the analysis of fuzzy systems is to replace fuzzy sets involved by their equivalent α-cuts (α-level sets). If A is a fuzzy set in X, then its α-cut (α-level set) is

$$A_\alpha = \{x \in X : A(x) \geq \alpha\}, \qquad \forall \alpha \in [0,1] \tag{16}$$

The α-cuts (α-level sets) may also be employed to obtain an equivalent of a fuzzy optimization problem as proposed in the classic Orlovski's approach [, (Orlovski 1977)] which, as opposed to a satisfaction based approach sketched in the former sections, is more explicitly related to optimization.

The problem considered is again as schematically shown in (1), i.e.

$$\widetilde{\max}_{x \tilde{\in} C} f(x) \tag{17}$$

where, similarly as in (1), the maximization and inclusion should be meant in a fuzzy way. Notice that, for simplicity, we assume here a nonfuzzy objective function $f(x)$ instead of a fuzzy one $F(x)$ as in (1).

First, for the fuzzy feasible set C we derive its α-cuts, $C_\alpha = \{x \in X : C(x) \geq \alpha\}$, for each $\alpha \in (0,1]$. Then, for each $\alpha \in (0,1]$ such that $C_\alpha \neq \emptyset$, we introduce the following (nonfuzzy) set

$$N(\alpha) = \{x \in X : f(x) = \sup_{x \in C_\alpha} f(x)\} \tag{18}$$

Now, a so-called *solution 1* to problem (17) is defined as the following fuzzy set [, (Orlovski 1977)]

$$S_1(x) = \begin{cases} \sup_{x \in N(\alpha)} \alpha & \text{for } x \in \cup_{\alpha > 0} N(\alpha) \\ 0 & \text{otherwise} \end{cases} = \\ = \begin{cases} C(x) & \text{for } x \in \cup_{\alpha > 0} N(\alpha) \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

and the fuzzy maximal value of $f(x)$ over the fuzzy feasible set C is defined as, for each $r \in R$:

$$f(r) = \sup_{x \in f^{-1}(r)} S_1(x) = \sup_{x \in f^{-1}(r)} \sup_{x \in N(\alpha)} \alpha \tag{20}$$

Next, a so-called *solution 2* to problem (17) is introduced [, (Orlovski 1977)]. For $f(x)$ and $C(x)$ we first define the set of Pareto maximal elements, P, as a (nonfuzzy) subset $P \subseteq X$ such that $x \in P$ if and only if there exists no $y \in X$ for which

$$\begin{cases} \text{either} \\ f(y) > f(x) \text{ and } C(y) \geq C(x) \\ \text{or} \\ f(y) \geq f(x) \text{ and } C(y) > C(x) \end{cases} \tag{21}$$

Then, solution 2 is defined as a fuzzy set

$$S_2(x) = \begin{cases} C(x) & \text{for } x \in P \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

It may then be shown [, (Orlovski 1977)] that this solution yields the same fuzzy maximal value of $f(x)$ over C as solution 1, i.e. for each $r \in R$:

$$f(r) = \sup_{x \in f^{-1}(r)} S_1(x) = \sup_{x \in f^{-1}(r)} \sup_{x \in N(\alpha)} \alpha \tag{23}$$

As to more interesting properties of solution 2, one may mention here $P \subset \bigcup_{\alpha>0} N(\alpha)$ which implies $S_2(x) \leq S_1(x)$, for each $x \in X$, i.e. solution 2 is a subset (in the sense of inclusion of two fuzzy sets) of solution 1.

Among other more relevant approaches in which $f(x)$ and $C(x)$ are dealt with separately ane may mention those by [, (Negoita and Ralescu 1977)] or [, (Yager 1979)].

3 Fuzzy mathematical programming

As already mentioned in Section 1, mathematical programming is meant as a special optimization problem in which the feasible set is given as (a set of) equalities and/or inequalities. For our purposes, a general mathematical programming problem can be written as

$$\begin{cases} \max_{x \in R^n} f(x) \\ \text{subject to: } g_i(x) \leq b_i; i = 1, \ldots, m \end{cases} \tag{24}$$

where $x = [x_1, \ldots, x_n]^T \in R^n$ is a vector of decision variables, $f : R^n \longrightarrow R$ is an objective function, $g_i : R^n \longrightarrow R$ are constraints, and $b_i \in R$ are the so-called right-hand sides; clearly, the maximization may readily be replaced by minimization, and "$\leq$" by "$\geq$".

Particularly important in practice is *linear programming* in which both the objective function and constraints are linear functions, i.e. (24) becomes

$$\begin{cases} \max_{x_j \geq 0} f(x) = cx = \sum_{j=1}^{n} c_j x_j \\ \text{subject to: } (Ax)_i = \sum_{j=1}^{n} a_{ij} x_j \leq b_i; i = 1, \ldots, m \end{cases} \tag{25}$$

and a large part of our next discussions will be devoted to fuzzy linear programming.

4 Fuzzy linear programming

In the context of fuzzy mathematical programming, fuzzy linear programming is also the most relevant from a practical point of view, and almost all more relevant developments in the field of fuzzy mathematical programming occured in the linear case. In this paper, too, emphasis will be on fuzzy linear programming.

Looking at the general linear programming problem formulation (25), we may readily point out the following elements that may be fuzzified:

- the coefficients (costs) in the objective function $f(x)$, i.e. $c = [c_1, \ldots, c_m]$,
- the coefficients a_{ij} in the so-called technological matrix $A = [a_{ij}]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and
- the right-hand sides $b = [b_1, \ldots, b_m]^T$

The above leads to the following basic types of fuzzy linear programming:

- problems with fuzzy constraints,
- problems with a fuzzy objective function (fuzzy goal),
- problems with fuzzy costs c_i's, and
- problems with fuzzy coefficients a_{ij}'s and b_i's,

which will be briefly presented below.

4.1 Fuzzy linear programming with fuzzy constraints

In this case the fuzzy linear programming problem may be generally written as:

$$\begin{cases} \max_{x \in R^n} cx \\ \text{subject to: } Ax \tilde{\leq} b; x \geq 0 \end{cases} \tag{26}$$

where "$\tilde{\leq}$" denotes an imprecise "less than" relation meant as that the left-hand side should be essentially less than or equal to the right-hand side with the understanding that this should be possibly well satisfied.

Usually, "$\tilde{\leq}$" is formalized by allowing the i-th constraint in (26) to be violated to some extent which is done by introducing a degree of satisfaction of the i-th constraints given as the following membership function

$$i(x) = \begin{cases} 1 & \text{if } (Ax)_i < b_i \\ h_i[(Ax)_i] & \text{if } b_i \leq (Ax)_i \leq b_i + t_i \\ 0 & \text{if } (Ax) > b_i + t_i \end{cases} \tag{27}$$

where $h_i(.) \in (0,1)$ is such that the higher the violation of the i-th constraint the lower the value of h_i. In practice, $h_i(.)$ is assumed to be a linear function, also in this paper. Moreover, t_i is the maximum vialation of the i-th constraint.

The first method for solving problem (26) is due to [, (Tanaka, Okuda and Asai 1974)] who show that the solution of the original problem (26) may be replaced by: find an optimal pair $(\alpha^*, x^*) \in [0,1] \times R^n$ such that

$$\alpha^* \wedge f(x^*) = \sup_{\alpha \in [0,1]} [\alpha \wedge \max_{x \in X_\alpha} f(x)] \tag{28}$$

where $f : R^n \longrightarrow [0,1]$ is a continuous objective functions, and $X_\alpha = \{x \in R^n \mid \bigwedge_{i=1,\ldots,m} i(x) \geq \alpha\}$, for each $\alpha \in (0,1]$.

As shown in [, (Tanaka, Okuda and Asai 1974)], under some mild assumptions concerning the continuity of the objective function f and the uniqueness of α^*, an optimal solution sought – (α^*, x^*) – is obtained using the following iterative algorithm:

Step 1: Assume $k = 1$ and an $\alpha_1 \in (0,1]$.
Step 2: Compute $f_k = \max_{x \in X_{\alpha_k}} f(x)$.
Step 3: Compute $\varepsilon_k = \alpha_k - f_k$. If $\mid \varepsilon_k \mid > \varepsilon$, then go to **Step 4**, otherwise go to **Step 5**; $\varepsilon \in [0,1]$ is a required precision.
Step 4: Compute $\alpha_{k+1} = \alpha_k - r_k \varepsilon_k$, where $r_k \geq 0$ is selected so that $0 \leq \alpha_{k+1} \leq 1$. Set $k := k+1$ and go to **Step 2**.
Step 5: Let $\alpha^* = \alpha$ and find an optimal $x^* \in R^n$ such that

$$f(x^*) = \max_{x \in X_{\alpha^*}} f(x) \tag{29}$$

Another approach to the solution of problem (26) is due to [, (Zimmermann 1976)]. Basically, his line of reasoning is inspired by the concept of a maximizing decision (8). First, by putting $e_j = -c_j$ into the objective function in problem (28), the maximization is replaced by the minimization. Then, problem (26) is replaced by its following fuzzified version

$$\begin{cases} ex = \sum_{j=1}^{n} e_j x_j \tilde{\leq} z & \\ Ax = \sum_{j=1}^{n} a_{ij} x_j \tilde{\leq} b_i & \text{for } i = 1, \ldots, m \\ x_j \geq 0 & \text{for } j = 1, \ldots, n \end{cases} \tag{30}$$

which should be meant as follows: ex should be "essentially smaller than or equal to" an aspiration level z, and the constraints' left-hand sides, Ax, should be "essentially smaller than or equal to" b_i's; evidently, both should be satisfied as well as posible (to the highest possible extent).

The first step is a proper formalization of "$\tilde{\leq}$" which stands for "essentially smaller than or equal to". We start by introducing the $(m+1) \times n$ matrix $H = [h_{kj}]$ which is formed by adding to the original matrix $A = [a_{ij}]$ the row vector $[e_j]$ before the first row of A.

We donote now the k-th row of Hx, the product of matrix H and vector x, by

$$(Hx)_k = \sum_{j=1}^{n} h_{kj} x_j \tag{31}$$

and define the function

$$g_k[(Hx)_k] = \begin{cases} 1 & \text{for } (Hx)_k \leq w_k \\ 1 - \frac{(Hx)_k - w_k}{t_k} & \text{for } w_k < (Hx)_k \leq w_k + t_k \\ 0 & \text{for } (Hx)_k \leq w_k + t_k \end{cases} \tag{32}$$

where $w = [w_1, \ldots, w_{m+1}]^T = [z, b_1, \ldots, b_m]^T$, and the t_k's are some admissible violations of the respective constraints.

Therefore, the function (32) does model an aspiration-level-based (degree of) satisfaction of the fuzzy goal and fuzzy constraints because its value is equal 1 if they are (perfectly) satisfied, dimishes as the degree of violation increases, and is 0 for the inadmissible violation, i.e. more than the t_k's.

In problem (30) we wish to satisfy all the constraints, and this can be expressed by the following objective function, being evidently a fuzzy decision in the sense of (3):

$$D(x) = \bigwedge_{k=1}^{m+1} g_k[(Hx)_k] \tag{33}$$

We wish to satisfy all the constraints to the highest possible extent [cf. the optimal decision (4)], i.e. we seek an optimal $x^* \in R^n$ such that

$$D(X^*) = \sup_{x \in R^n} D(x) \tag{34}$$

It may be shown [cf. [, (Negoita and Ralescu 1977)] or [, (Zimmermann 1976)], though this result was known earlier] that each optimal solution, (λ^*, x^*), of the following linear programming problem

$$\begin{cases} & \max_{\lambda \in (0,1]} \lambda \\ \text{subject to:} & \lambda \leq w'_k - (Hx)'_k \;\; k = 1, \ldots, m+1 \\ & x_j \geq 0 \qquad\qquad\quad j = 1, \ldots, n \end{cases} \tag{35}$$

where $w'_k = \frac{w_k}{t_k}$ and $(Hx)'_k = \frac{(Hx)_k}{t_k}$, $k = 1, \ldots, m+1$, is also an optimal solution to problem (34).

The third basic approach to the solution of problem (26) was proposed in [, (Verdegay 1982)]. It employs the so-called *representation theorem* [, (Negoita and Ralescu 1975)] which basically says that a fuzzy set can be uniquely represented by all its α-cuts; this theorem can be found in any book on fuzzy sets.

First, if the membership functions of the fuzzy constraints in problem (26) are strictly monotone and continuous, which is often natural, then the α-cuts of the set of constraints can be expressed by

$$C_\alpha = \{x \in R^n \mid \sum_{j=1}^{n} a_{ij} x_j \leq g^{-1}(\alpha); x_k \geq 0; i = 1, \ldots, m; j = 1, \ldots, m\} \tag{36}$$

where the $g_i^{-1}(\alpha)$'s are the inverse functions of the $g_i(.)$'s defined by (27).

Then, if C denotes the set of fuzzy constraints in problem (26), the representation theorem states that

$$C = \sum_{\alpha \in (0,1]} \alpha C_\alpha \tag{37}$$

A fuzzy solution (for all $\alpha \in (0,1]$!) to problem (26) can be therefore obtained by solving the following parametric linear programming problem

$$\begin{cases} \max_{x \in R^n} cx \\ \text{subject to: } x \in C_\alpha; \text{for each } \alpha \in (0,1] \end{cases} \quad (38)$$

or, more explicitly:

$$\begin{cases} \max_{x \in R^n} cx \\ \text{subject to: } (Ax)_i \leq g_i^{-1}(\alpha); \text{for each } \alpha \in (0,1]; i = 1,\dots,m \end{cases} \quad (39)$$

And for linear fuzzy constraints, which is practically the most interesting case, problem (39) becomes

$$\begin{cases} \max_{x \geq 0} cx \\ \text{subject to: } Ax \leq b + t(1-\alpha); \text{for each } \alpha \in (0,1] \end{cases} \quad (40)$$

where $t = [t_1,\dots,t_m]^T$ is a vector of admissible violations of the particular constraints.

Thus, if $x^*(\alpha)$ is an optimal solution to problem (39), from it (for each $\alpha \in (0,1]$) a fuzzy optimal solution to problem (37) can be obtained.

Finally, it is interesting to notice [cf. [, (Verdegay 1982)]] that if we denote by $f(x)$ the objective function in problem (28), and by $f_{m+1}(x)$ the membership function of the fuzzy goal in problem (32), then the solution of problem (26) derived by employing the approach proposed in [, (Tanaka, Okuda and Asai 1974)], and its corresponding solution of (32) derived by employing the one in [, (Zimmermann 1976)], can be obtained from the fuzzy solution of problem (39), $x^*(\alpha)$, by solving the following equations, for a given $\alpha \in (0,1]$:

$$f[cx^*(\alpha)] = \alpha \quad f_{m+1}[cx^*(\alpha)] = \alpha \quad (41)$$

Notice that while solving problem (26) by using the linear programming problem (39) it is necessary to define $f(.)$ and $f_{m+1}(.)$ a priori. Moreover, the size of problem (39) is as that of (26).

Finally, notice that if the $g_i(.)$'s in (36) are nonlinear, this nonlinearity occurs in the right-hand sides of the constraints in problem (39) but it does not imply the nonlinearity of the auxiliary parametric linear program (40), so that problem (39) is also valid for nonlinear constraints.

4.2 Fuzzy coefficients in the objective function

In this section we will consider fuzzy mathematical programming problems in which the constraints are nonfuzzy and the coefficients in the objective function, c_j, are fuzzy numbers given by the membership functions $j : R^n \longrightarrow [0,1]$, $j = 1,\dots,n$. This class of problems may be written as

$$\begin{cases} \max_{x \in R; x \geq 0} cx \\ \text{subject to: } Ax \leq b \end{cases} \quad (42)$$

where c_j, $j = 1,\dots,n$, are assumed to be fuzzy.

We will sketch three approaches to the formulation and solution of such problems which are due to: [, (Delgado, Verdegay and Vila 1987b)], [, (Tanaka, Ichihashi and Asai 1984)], and [, (Rommelfanger, Hanuscheck and Wolf 1989)]. For yet another approach we will refer the reader to [, (Chanas and Kuchta 1994)].

In [, (Delgado, Verdegay and Vila 1987b)], the coefficients in the objective function of problem (42), $c = [c_1,\dots,c_j,\dots,c_n]$, are assumed to be fuzzy sets (numbers) such that, for c_j:

$$j(x) = \begin{cases} 0 & \text{if } \overline{c}_j \le x \text{ or } x \le \underline{c}_j \\ \underline{h}_j(x) & \text{if } \underline{c}_j < x \le c_j \\ \overline{h}_j(x) & \text{if } c_j < x \le \overline{c}_j \end{cases} \tag{43}$$

where $[\overline{c}_j, \underline{c}_j]$ is the support of the fuzzy number c_j, and $\underline{h}_j(.)$ and $\overline{h}_j(.)$ are continuous and strictly increasing and decreasing, respectively, functions such that $\underline{h}_j(c_j) = \overline{h}_j(c_j) = 1$.

Then, using a fuzzy objective function as defined in [, (Verdegay 1982)] and including the $(1-\alpha)$-cuts of each cost coefficient, for each $\alpha \in (0,1]$, we have for each $x \in R$ and for $j = 1,\dots,n$

$$j(x) \ge 1-\alpha \Longleftrightarrow \underline{h}_j^{-1}(1-\alpha) \le x \le \overline{h}_j^{-1}(1-\alpha) \tag{44}$$

and if we denote $= \Phi_j(.) = \underline{(h)}_j(.)$ and $\Psi(.) = \overline{h}_j(.)$, then we obtain

$$\Phi_j(1-\alpha) \le x \le \Psi_j(1-\alpha), \qquad \text{for each } x \in R \tag{45}$$

As shown in [, (Verdegay 1982)], a fuzzy solution to problem (42) can be found from the parametric solution of the following multiobjective linear program

$$\begin{cases} & \max_{x \in R; x \ge 0} \{c^1 x, c^2 x, \dots c^{2^n} x\} \\ \text{subject to:} & Ax \le b \end{cases} \tag{46}$$

where $c^k \in E(1-\alpha)$, $\alpha \in (0,1]$, $k = 1,2,\dots,2^n$, and $E(1-\alpha)$ is the set of vectors in R^n each of whose components is either on the upper bound, $\Psi_j(1-\alpha)$, or on the lower bound, $\Phi_j(1-\alpha)$, of the respective $(1-\alpha)$-cuts.

On the other hand, in [, (Tanaka, Ichihashi and Asai 1984)] a possibilistic approach is proposed for even more general problem formulations, with fuzzy constraints in addition to fuzzy coefficients in the objective function as in (42). Their method is based on the two relevant facts. First, if the c_j's, $j = 1,\dots,n$, are triangular fuzzy numbers with membership function written as $\delta(\underline{c}_j, c_j, \overline{c}_j)$, then the value of the objective function, $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$, is also a fuzzy number with the membership function

$$G(y) = \begin{cases} \frac{1-(|2y-(\overline{c}+\underline{c})|x)}{(\overline{c}-\underline{c})x} & \text{for } x > 0, y > 0 \\ 1 & \text{for } x = 0, y > 0 \\ 0 & \text{for } x = 0, y = 0 \end{cases} \tag{47}$$

where $\overline{c} = [\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_n]$, $\underline{c} = [\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_n]$, and the $\underline{c}_j$'s and $\overline{c}_j$'s are defined as in (43).

The second assumption is that the maximization in (42) concerns now a fuzzy function, which may be written as $\widetilde{\max}_{x \in R; xge0}$, and this is meant as

$$\max_{x \in R; x \geq 0} (w_1 \overline{c} x + w_2 \underline{c} x)$$

where $w_1, w_2 \in [0,1]$, $w_1 + w_2 = 1$, are some weights.

Then, the solution of problem (42) is obtained by solving the following auxiliary linear programming problem

$$\begin{cases} \max_{x \in R; x \geq 0} (w_1 \overline{c} x + w_2 \underline{c} x) \\ \text{subject to:} \quad Ax \leq b \end{cases} \tag{48}$$

The solution of problem (42) was also discussed by [, (Rommelfanger, Hanuscheck und Wolf 1989)]. Their approach is termed a *stratified piecewise reduction* approach, and is based in its essence on the classic approach from [, (Zimmermann 1976)]. Though the problem is given as (42), the imprecision (fuzziness) of coefficients is modelled by the use of nested intervals. Each of such intervals, say interval k, is assigned a membership degree, or a possibility degree $\alpha_k \in [0,1]$, $k = 1, 2, \ldots, p$, where p is the number of such intervals. Then, each fuzzy coefficient c_j, $j = 1, 2, \ldots, n$, in the objective function of (42) is defined as the fuzzy set

$$c_j = \frac{[\underline{c}_j, \overline{c}_j]^k]}{\alpha_k}, \qquad k = 1, 2, \ldots, p \tag{49}$$

such that, for all $\alpha_1, \alpha_2 \in [0,1]$, and $j = 1, 2, \ldots, n$, there holds

$$\alpha_1 \geq \alpha_2 \Longrightarrow [\underline{c}_j, \overline{c}_j]^1 \subseteq [\underline{c}_j, \overline{c}_j]^2 \tag{50}$$

Clearly, fuzzy coefficients given as (43) may also be equivalently formulated by using α-cuts as shown by (49).

Then, the solution of problem (42) is obtained by solving the following auxiliary linear programming problem [cf. [, (Rommelfanger, Hanuscheck and Wolf 1989)]]

$$\begin{cases} \max_{\lambda \in (0,1]} \lambda \\ \text{subject to:} \quad f_1 [\underline{c}^\alpha x] \geq \lambda \\ \qquad f_2 [\overline{c}^\alpha x] \geq \lambda \\ \qquad Ax \leq b; x \geq 0 \end{cases} \tag{51}$$

where:

- if $\overline{z}^\alpha_{\min} \leq \underline{c}^\alpha x \leq z^{*\alpha}_{\min}$, then

$$f_1 [\underline{c}^\alpha] = \frac{\underline{c}^\alpha x - \overline{z}^\alpha_{\min}}{z^{*\alpha}_{\min} - \overline{z}^\alpha_{\min}}$$

- if $\overline{z}^{\alpha}_{\cdot\max} \leq \overline{c}^{\alpha} x \leq z^{*\alpha}_{\cdot\max}$, then

$$f_2[\overline{c}^{\alpha}] = \frac{\overline{c}^{\alpha} x - \overline{z}^{\alpha}_{\cdot\max}}{z^{*\alpha}_{\cdot\min} - \overline{z}^{\alpha}_{\cdot\max}}$$

where, if we denote $\mathscr{X} = \{x \in R^n : Ax \leq b, x \geq 0\}$, then:

$$\begin{array}{l} z^{*\alpha}_{\cdot\min} = \underline{c}^{\alpha}(x^*_{\min}) = \max\{\underline{c}^{\alpha} x \mid x \in \mathscr{X}\} \\ z^{*\alpha}_{\cdot\max} = \overline{c}^{\alpha}(x^*_{\max}) = \max\{\overline{c}^{\alpha} x \mid x \in \mathscr{X}\} \\ \overline{z}^{\alpha}_{\cdot\min} = \underline{c}^{\alpha}(x^*_{\max}) = \min\{\underline{c}^{\alpha} x \mid x \in \mathscr{X}\} \\ \overline{z}^{\alpha}_{\cdot\min} = \overline{c}^{\alpha}(x^*_{\min}) = \min\{\overline{c}^{\alpha} x \mid x \in \mathscr{X}\} \end{array}$$

The solution of problem (42) may be then found as the intersection of solutions obtained by solving problem (51) for each $\alpha_k \in, k = 1, 2, \ldots, p$.

To conclude the description of this approach, notice that problem (51) may be rewritten as

$$\left\{ \begin{array}{ll} & \max_{\lambda \in (0,1]} \lambda \\ \text{subject to:} & (z^{*\alpha}_{\cdot\min} - \overline{z}^{\alpha}_{\cdot\min})\lambda - \underline{c}^{\alpha} x \leq \overline{z}^{\alpha}_{\cdot\min} \\ & f_2[\overline{c}^{\alpha} x] \geq \lambda \\ & (z^{*\alpha}_{\cdot\max} - \overline{z}^{\alpha}_{\cdot\max})\lambda - \underline{c}^{\alpha} x \leq \overline{z}^{\alpha}_{\cdot\max} \\ & Ax \leq b; x \geq 0 \end{array} \right. \tag{52}$$

Clearly, $f_1[\underline{c}^{\alpha} x^*] = f_2[\overline{c}^{\alpha} x^8] = \lambda^*$, where (λ^*, x^*) is an optimal solution of problem (52).

Finally, it should be mentioned that problems (46), (48) and (52) are closely interrelated which will not be however elaborated upon here, and we will refer the interested reader to, e.g., [, (Delgado, Verdegay and Vila 1990)].

4.3 Fuzzy coefficients in the technological matrix

In this case the coefficients in problem (42) in the so-called technological matrix, A, and in the right-hand sides, b, are fuzzy numbers assumed, for simplicity, to be in the L-R form [cf. [, (Dubois and Prade 1980)]]. On the other hand, the coefficients in the objective function, c, are nonfuzzy (i.e. real numbers).

In this case the fuzzy linear programming problem is written as

$$\left\{ \begin{array}{ll} & \max_{x \in R; x \geq 0} cx \\ \text{subject to:} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, \ldots, m \end{array} \right. \tag{53}$$

This class of problems was discussed in [, (Tanaka, Ichihashi and Asai 1984)]; in fact, they also dicussed the case of fuzzy coefficients in the objective function but this case will not be considered here for simplicity.

First, it should be remarked that in problem (53) the fuzziness is in the coefficients, and not in the allowable violation of the constraints. Therefore, problem (52) is principally different than problem (26) with a fuzzy constraint set.

Basically, for solving problem (53), in [, (Tanaka, Ichihashi and Asai 1984)] the following auxiliary conventional linear programming problem is solved

$$\begin{cases} & \max_{x \in R; x \geq 0} cx \\ \text{subject to:} & [(1-\frac{\beta}{2})(\underline{a}_i + a_i) + \frac{\beta}{2}(a_i - \underline{a}_i)]x \leq \\ & \leq (1-\frac{\beta}{2})(b_i + \underline{b}_i) + \frac{\beta}{2}(b_i - \underline{b}_i) \\ & [\frac{\beta}{2}(\overline{a}_i + a_i) + (1-\frac{\beta}{2})(\overline{a}_i - a_i)]x \leq \\ & \leq \frac{\beta}{2}(b_i + \overline{b}_i) + (1-\frac{\beta}{2})(\overline{b}_i - b_i) \end{cases} \tag{54}$$

where $\beta \in [0,1]$ is a degree of optimism to be specified a priori.

As a prerequisite for obtaining the auxiliary linear programming problem (53), the following ordering relation between the triangular fuzzy numbers is assumed

$$a \tilde{>}_\beta b \Longleftrightarrow (a+\overline{a})_k \geq (b+\overline{b})_k \quad \& \quad (a-\underline{a})_k \geq (b-\underline{b})_k \tag{55}$$

for each $k \in [\beta, 1]$, where $(a+\overline{a})_k$ and $(a-\underline{a})_k$ are the upper and lower bounds, respectively, of the k-cut of a.

Notice that in (55) no fuzziness in respect to the satisfaction of constraints is involved. This is accounted for in a model of fuzzy linear programming proposed in [, (Delgado, Verdegay and Vila 1989)]. Its point of departure is the following problem

$$\begin{cases} & \max_{x \in R; x \geq 0} cx \\ \text{subject to:} & \sum_{j=1}^{n} a_{ij} x_j \tilde{\leq} b_i; i = 1, \ldots, m \end{cases} \tag{56}$$

where "$\tilde{\geq}$" means that some violation of "$\geq$" may be allowed; as before, the a_{ij}'s and b_j's are fuzzy, given as triangular fuzzy numbers.

The above violation, for the i-th constraint, is expressed by a fuzzy number t_i, a margin of violation tolerance. Then, the set of constraints in (56) is replaced by

$$\sum_{j=1}^{n} a_{ij} x_j > b_i + t_i(1-\alpha), \qquad \alpha \in [0,1], i = 1, \ldots, m \tag{57}$$

where "$>$" is some relation between two fuzzy numbers, preserving only the ranking under the multiplication by a positive scalar.

Therefore, the problem considered becomes

$$\begin{cases} & \max_{x \in R; x \geq 0} cx \\ \text{subject to:} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i + t_i(1-\alpha) \\ & \alpha \in [0,1], i = 1, \ldots, m \end{cases} \tag{58}$$

from which a fuzzy optimal solution of (56) can be obtained.

4.4 Remarks on duality in fuzzy linear programming

Duality is an important issue in linear programming, and it has also attracted attention in fuzzy liner programming. The first approach is presumably due to [, (Hamacher, Leberling and Zimmermann 1978)]. They start from a (mixed) fuzzy linear programming problem with both fuzzy and nonfuzzy constraints. Its solution is obtained by solving an auxiliary (nonfuzzy) linear programming problem of type (35). Its dual is obtained, and then the dual variables are analyzed and interpreted.

In another approach due to Verdegay [, (Verdegay 1984)], which takes advantage of the symmetry between the fuzzy constraints and fuzzy objective, the two types of fuzzy linear problems are considered:

- with a nonfuzzy objective function and fuzzy constraints, i.e. problem (26), and
- with nonfuzzy constraints and an objective function with fuzzy coefficients, i.e. problem (42).

For the former problem (26), i.e.

$$\begin{cases} \max_{x \in R^n} cx \\ \text{subject to: } Ax \tilde{\leq} b; x \geq 0 \end{cases} \tag{59}$$

in which "$\tilde{\leq}$" is formalized by allowing the i-th constraint in (59) to be violated to some extent which is done by introducing the degree of satisfaction of the i-th constraints given as [cf. (27)]

$$i(x) = \begin{cases} 1 & \text{if } (Ax)_i < b_i \\ h_i[(Ax)_i] & \text{if } b_i \leq (Ax)_i \leq b_i + t_i \\ 0 & \text{if } (Ax)_> b_i + t_i \end{cases} \tag{60}$$

it can be shown [cf. [, (Verdegay 1984)]] that the dual problem to (59) is

$$\begin{cases} \min_{u \in R} du \\ \text{subject to: } uA^T \geq c; u \geq 0 \end{cases} \tag{61}$$

which is a fuzzy linear programming problem with fuzzy cofficients d specified by (60).

As shown in [, (Verdegay 1984)], both the problems (59) and (61) have, for each α-cut, $\alpha \in (0,1]$, the same fuzzy solution which can be obtained using auxiliary parametric problems of type (36) and (46), respectively.

It can also be shown that in a similar way one can obtain dual problems for fuzzy linear programming problems with both fuzzy constraints and fuzzy coefficients in the objective functions [, (Verdegay 1984)].

5 Fuzzy nonlinear programming

Nonlinear programming is a wide class of mathematical programming problems of the type (24), i.e.

$$\begin{cases} & \max_{x \in R^n} f(x) \\ \text{subject to:} & g_i(x) \leq b_i; i = 1, \ldots, m \end{cases} \tag{62}$$

in which the objective function $f(x)$ and the constraints $g_i(x)$, $i = 1, \ldots, m$, are nonlinear as opposed to being linear as in Section 4.

Since that nonlinearity can enter problem (62) in defferent forms, this gives rise to a wide array of nonlinear programming problems exemplified by: quadratic programming, integer programming, 0–1 (Boolean) programming, geometric programming, etc. For details we refer the reader to any book on mathematical programming, operations research, etc. available from any major scientific publisher.

For most of the above mentioned nonlinear programming problems their fuzzifications have been proposed. For clarity and lack of space we will briefly present here fuzzy integer and 0–1 programming only, and give a brief overview of other types. We will show the corresponding fuzzy mathematical programming problems, and indicate their solutions. We will not discuss basic properties of fuzzy nonlinear programs – cf. [, (Diamond and Kloeden 1994)] for some details in this respect.

5.1 *Fuzzy integer programming*

Variables which take on integer values are of utmost importance in many problems in which they stand for countable entities as, e.g., numbers of people, parcels, cars, etc. Clearly, such problems abound in practice.

The same is true for variables which take on two values only, either 1 or 0. They may represent, for 1, say, the performance of a job on a machine, the inclusion of a container in a freight, etc. And the opposite for 0.

The first successful attempt at formulating and solving fuzzy integer programming problems is presumably due to Fabian and Stoica [, (Fabian and Stoica 1984)]. They start with the conventional integer programming problem which may be written as

$$\begin{cases} & \max_{x \in R} f(x) \\ \text{subject to:} & g(x) \leq 0 \\ & x = [x_1, \ldots, x_n], x_i \geq 0 \\ & x_i - \text{integers}, i = 1, \ldots, n \end{cases} \tag{63}$$

where $f(x)$ and $g(x)$ are real-valued functions.

Problem (63) is then fuzzified as follows:

$$\begin{cases} & \widetilde{\max}_{x \in R} f(x) \\ \text{subject to:} & g(x) \widetilde{\leq} 0 \\ & x = [x_1, \ldots, x_n], x_i \geq 0 \\ & x_i - \widetilde{\text{-integers}}, i = 1, \ldots, n \end{cases} \tag{64}$$

which should be read that a "possibly maximal" ($\widetilde{\max}$) solution x^* is sought which satisfies the constraints to a "possibly high" degree ($\widetilde{\leq}$), and whose components x_i's are "almost integer" ($\widetilde{\text{integers}}$).

Basically, the "almost integer" numbers are represented by very "narrow" (with very small left and right spreads) triangular fuzzy numbers, and the fuzziness in the concept of a maximal solution and constraint satisfaction is represented similarly as in Section 4.1. An equivalent nonlinear mixed integer program is derived and a solution procedure is given.

The model proposed in [, (Fabian and Stoica 1984)] has found applications in production scheduling.

A solution technique for solving fuzzy integer programming models with multiple criteria was proposed in [, (Ignizio and Daniels 1983)].

Another approach to integer fuzzy linear programming is due to [, (Herrera and Verdegay 1991)]. They have developed fuzzy integer programming models with fuzzy coefficients in the objective function and fuzzy coefficients in the technological matrix which are counterparts of those presented in the previous section. A good source of information on these topics is also [, (Herrera 1994)].

All the models mentioned above cover the cases of both a pure and mixed integer fuzzy linear programming, i.e with all integer variables and both integer and real variables, rspectively.

5.2 *Fuzzy 0–1 programming*

Variables which take on two values only, either 1 or 0, are very convenient in many situations. For instance, they may represent, in the case of 1, the performance of a job on a machine, the inclusion of a container in a freight, etc. And the opposite in the case of 0.

In spite of utmost importance of 0–1 mathematical programming problems, attempts at their fuzzification are quite rare, and have not attracted much attention. The most prominent early works in this area are [, (Zimmermann and Pollatschek 1979)] and [, (Zimmermann and Pollatschek 1984)]. For a newer analysis of approaches and solution techniques, we refer the reader to [, (Herrera and Verdegay 1991)].

The work [, (Zimmermann and Pollatschek 1984)] concerns the basic Zimmermann's fuzzy linear programming model [, (Zimmermann 1976)] [cf. (30)].

The 0–1 fuzzy linear programming problem is formulated as

$$\begin{cases} ex = \sum_{j=1}^{n} e_j x_j \tilde{\leq} z \\ Ax = \sum_{j=1}^{n} a_{ij} x_j \tilde{\leq} b_i & \text{for } i = 1, \ldots, m \\ x_j \in \{0, 1\} & \text{for } j = 1, \ldots, n \end{cases} \tag{65}$$

which should be meant similarly as (30).

Then, following the line of reasoning analogous to (31)–(35), an equivalent nonfuzzy 0–1 linear programming problem is obtained and a branch-and-bound procedure for its solution is proposed.

Another approach to integer fuzzy 0–1 programming is due to Herrera and Verdegay [, (Herrera and Verdegay 1991)]. They have developed fuzzy 0–1 programming models with fuzzy coefficients in the objective function and fuzzy coefficients in the technological matrix which are counterparts of the fuzzy linear programming problems presented in Section 4.

All the models mentioned above cover the cases of both a pure and mixed 0–1 fuzzy linear programming.

Among most relevant later works on fuzzy 0–1 programming one should mention [Castro, Herrera and Verdegay 1992, (Castro, Herrera and Verdegay 1992)]. A good source of information on these topics is also [, (Herrera 1994)].

5.3 *Remarks on other types of fuzzy nonlinear programming*

In addition to fuzzy integer and 0–1 programming problems described above, which are presumably the most relevant from the practical point of view, some other types on nonlinear fuzzy programming problems have been proposed in the literature.

Geometric programming, whose essence is that the objective functions and constraints are generalized polynomials, has found applications in many practical problems exemplified by engineering design, process planning, marketing management, etc. Its fuzzification has been proposed by [, (Cao 1987)], and then by [, (Verma 1990)]. We also refer the reader to [, (Sotirov and Mincoff 1994)] for more information, list of other contributions and some extensions.

Linear fractional programming is characterized by the objective function which is the fraction with both the numerator and denominator being linear functions. It has found many applications in diverse areas exemplified by production planning, financial and corporate planning etc. when the objective function may be, say, the ratio “inventory/sales” or “output/employee”, with both the terms given as linear functions of some decision variables. To an interesting case of multiple objective fractional programming we refer the interested reader to, e.g., [, (Sakawa and Yano 1994)].

For some other types of nonlinear programming problems, mainly of a combinatorial type, we refer the reader to the articles in the two edited volumes on fuzzy optimization [, (Kacprzyk and Orlovski 1987a)] and [, (Delgado, Kacprzyk, Verdegay and Vila 1994)].

6 Fuzzy dynamic programming

Dynamic programming, introduced in the mid-1950's by Bellman (cf. [, (Bellman 1957)], is a powerful solution technique for multistage optimization (control) problems which may be roughly characterized for our purposes as the ones in which there is some dynamics in the sense of time-varying goals, constraints, dynamic system under control, etc.

A fuzzification of dynamic programming appeared in the late 1960s in, e.g., [, (Bellman and Zadeh 1970)] works on fuzzy dynamic programming, and was then further developed – for an extensive exposition we refer the reader to Kacprzyk's books ([, Kacprzyk 1983], [, Kacprzyk 1997]).

The basic problem formulation may be stated as follows:

- the *system under control* (deterministic for now) is governed by its state transition equation
$$x_{t+1} = f(x_t, u_t), \qquad t = 0, 1, \ldots \tag{66}$$
where $x_t, x_{t+1} \in X$ are states at control stage t and $t+1$, and $u_t \in U$ is control at t, $t = 0, 1, \ldots, N-1$; N is the *termination time*,
- at each control stage t, $t = 0, 1, \ldots, N-1$, a *fuzzy constraint* on u_t, $C^t(u_t)$, and a *fuzzy goal* on x_{t+1}, $G^{t+1}(x_{t+1})$, are imposed,
- the performance of the control process is evaluated by [cf. [, (Bellman and Zadeh 1970)]] fuzzy decision [cf. (3)]
$$\begin{aligned} &D(u_0, \ldots, u_{N-1} \mid x_0) = \\ &= C^0(u_0) \wedge G^1(x_1) \wedge \ldots \wedge C^{N-1}(u_{N-1}) \wedge G^N(x_N) = \\ &= \bigwedge_{t=0}^{N-1} [C^t(u_t) \wedge G^{t+1}(x_{t+1})] \end{aligned} \tag{67}$$
- we seek an *optimal sequence of controls* $u_0^*, \ldots, u_{N-1}^*$ such that [cf. (4)]
$$\begin{aligned} &D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ &= \max_{u_0, \ldots, u_{N-1}} D(u_0, \ldots, u_{N-1} \mid x_0) = \\ &= \max_{u_0, \ldots, u_{N-1}} \bigwedge_{t=0}^{N-1} [C^t(u_t) \wedge G^{t+1}(x_{t+1})] \end{aligned} \tag{68}$$

Problem (68) leads to various classes, with a convenient classification with respect to [, (Kacprzyk 1997)]:

- type of the termination time: (a) fixed and specified in advance, (b) implicitly given (by entering a termination set of states), (c) fuzzy, and (d) infinite;
- type of the system under control: (a) deterministic, (b) stochastic, and (c) fuzzy.

In this short review we will mainly outline the case of a fixed and specified termination time, and the cases of a deterministic, stochastic and fuzzy systems under control, with emphasis on the deterministic one. For other cases and details we will refer the interested reader to [, (Kacprzyk 1997)].

The case of a fixed and specified termination time is basic and will be discusses in more detail. We start with the deterministic system under control given by its state transition equation (66). The problem considered is (68).

The two basic solution techniques are:

- dynamic programming, and
- branch-and-bound,

and the former will be discussed here. For the latter, and for the two newer approaches based on a neural network and genetic algorithm we refer the reader to [, (Kacprzyk 1997)].

In problem (68) we can readily notice that the two last right-hand side terms

$$C^{N-1}(u_{N-1}) \wedge G^N[f(x_{N-1}, u_{N-1})]$$

depend only on u_{N-1} and not on other controls.

Thus, $\max_{u_0,\dots,u_{N-1}}$ can be split into $\max_{u_0,\dots,u_{N-2}}$ and $\max_{u_{N-1}}$. And similarly, one can split $\max_{u_0,\dots,u_{N-2}}$ can be split into $\max_{u_0,\dots,u_{N-3}}$ and $\max_{u_{N-2}}$, etc.

This leeds to the set of recurrence equations yielding an optimal control policy $a_t^* : X \longrightarrow U$, $u_t = a_t^*(x_t)$:

$$\begin{cases} G^{N-i}(x_{N-i}) = \max_{u_{N-i}}[C^{N-i}(u_{N-i}) \wedge G^{N-i+1}(x_{N-i+1})] \\ x_{N-i+1} = f(x_{N-i}, u_{N-i}); i = 0, 1, \dots, N \end{cases} \tag{69}$$

The stochastic system under control is assumed to be a Markov chain whose dynamics (state transitions) is governed by a conditional probability function

$$p(x_{t+1} \mid x_t, u_t), \qquad t = 0, 1, \dots \tag{70}$$

which specifies the probability of attaining $x_{t+1} \in X = \{s_1, \dots, s_n\}$ from $x_t \in X$, under $u_t \in U = \{c_1, \dots, c_m\}$,

At each $t = 0, 1, \dots, N-1$, $u_t \in U$ is subjected to a fuzy constraint $C^t(u_t)$, and on $x_N \in X$, a fuzzy goal $G^N(x_N)$ is imposed.

The two different problem formulations are used:

- Bellman and Zadeh's formulation [, (Bellman and Zadeh 1970)]: find $u_0^*, \dots, u_{N-1}^*$ maximizing the probability of attainment of the fuzzy goal subject to the fuzzy constraints, i.e.

$$D(u_0^*, \dots, u_{N-1}^* \mid x_0) = \max_{u_0, \dots, u_{N-1}} [C^0(u_0) \wedge \dots$$
$$\dots \wedge C^{N-1}(u_{N-1}) \wedge EG^N(x_N)] \tag{71}$$

- Kacprzyk and Staniewski's formulation [, (Kacprzyk and Staniewski 1980)]: find $u_0^*, \dots, u_{N-1}^*$ maximizing the expected value of the fuzzy decision, i.e.

$$\begin{aligned}
&D(u_0^*,\ldots,u_{N-1}^* \mid x_0) = \\
&= \max_{u_0,\ldots,u_{N-1}} ED(u_0,\ldots,u_{N-1} \mid x_0) = \\
&= \max_{u_0,\ldots,u_{N-1}} E[C^0(u_0) \wedge \ldots \\
&\ldots \wedge C^{N-1}(u_{N-1}) \wedge G^N(x_N)]
\end{aligned} \tag{72}$$

and in both we evidently seek an optimal control policies $a_t^* : X \longrightarrow U$, such that $u_t^* = a_t^*(x_t), t = 0,1,\ldots,N-1$.

The probability of a fuzzy event in both the problem formulations (71) and (72) is meant in Zadeh's sense [, (Zadeh 1968)], i.e. as a real number in $[0,1]$.

Kacprzyk and Staniewski's formulation is much more difficult to solve, and somehow specific. In this paper we will employ the classic Bellman and Zadeh's formulation given as (71) which leads to dynamic programming recurrence equations.

First, the G^N is regarded as a fuzzy event in X, and the conditional probability of this event given x_{N-1} and u_{N-1} is expressed by

$$\begin{aligned}
&EG^N(x_N) = EG^N(x_N \mid x_{N-1},u_{N-1}) = \\
&= \sum_{x_N \in X} p(x_N \mid x_{N-1},u_{N-1}) \cdot G^N(x_N)
\end{aligned} \tag{73}$$

It may readily be noticed that the structure of problem (71) make the use of dynamic programming possible. Namely, $EG^N(x_N) = EG^N[f(x_{N-1},u_{N-1})]$, i.e. is a function of x_{N-1} and u_{N-1}. Therefore, the two right-most terms in the right-hand side of (71) depend on u_{N-1} and not on the other controls. The second right-most terms depend on u_{N-2}, etc. The maximization over $u_0,\ldots,u_{N-1}$ in (71) can be therefore split into the consecutive maximizations with respect to the particular u_t's, $t = N-1, N-2,\ldots,0$. This leads to the following set of dynamic programming recurrence equations:

$$\begin{cases}
G^{N-1}(x_{N-1}) = \max_{u_{N-1}} [C^{N-i}(u_{N-i}) \wedge EG^{N-i+1}(x_{N-i+1})] \\
EG^{N-1+1}(x_{N-i+1}) = \sum_{x_{N-i} \in X} p(x_{N-i+1} \mid x_{N-i},u_{N-i}) \times \\
\quad \times G^{N-i+1}(x_{N-i+1}) \\
i = 1,\ldots,N
\end{cases} \tag{74}$$

where $G^{N-i}(x_{N-i})$ may be viewed as a fuzzy goal at $t = N-i$ induced by $G^{N-i+1}(x_{N-i+1})$.

The successive maximizing values of u_{N-i}, u_{N-i}^*, $i = 1,2,\ldots,N$, give the optimal control policies a_{N-i}^* such that $u_{N-i}^* = a_{N-i}^*(x_{N-i})$, $i = 1,\ldots,N$.

The fuzzy system is governed by a fuzzy state transition equation

$$X_{t+1} = F(X_t,U_t), \qquad t = 0,1,\ldots \tag{75}$$

where X_t and X_{t+1} are fuzzy states at t and $t+1$, U_t is a fuzzy control at t, $t = 0,1,\ldots,N-1$.

The $\overline{C}^t(U_t)$ is a *fuzzy constraint* on U_t, and $\overline{G}^{t+1}(X_{t+1})$ is a *fuzzy goal* on X_{t+1}; $\overline{G}^{t+1}$ and $\overline{C}^t$ mean that the original G^{t+1} and C^t are modified to account for the fuzziness of the state and control (e.g., $\overline{G}^N(X_N) = 1 - \mathrm{d}(X_N, G^N)$; d(.,.) is some distance measure).

We seek an *optimal sequence of fuzzy controls* $U_0^*, \ldots, U_{N-1}^*$ such that

$$D(U_0^*, \ldots, U_{N-1}^* \mid X_0) = \\ = \max_{U_0,\ldots,U_{N-1}} \bigwedge_{t=0}^{N-1} [\overline{C}^t(U_t) \wedge \overline{G}^{t+1}(X_{t+1})] \tag{76}$$

and traditionally this problem is solved by dynamic programming [, (Baldwin and Pilsworth 1982)] and branch-and-bound [, (Kacprzyk 1979)].

In dynamic programming, after some "trickery" the set of recurrence equations of the type (68) is obtained. To make it practically solvable, a simple "natural" approach is used ([, Baldwin and Pilsworth 1982], [, Kacprzyk and Staniewski 1982]) in which some prespecified *reference fuzzy states* and *controls*, $\overline{U}_t \in \overline{\mathscr{U}} \subseteq \mathscr{U}$, and $\overline{X}_{t+1} \in \overline{\mathscr{X}} \subseteq \mathscr{X}$, are used, and we express all U_{t-1}'s and X_t's by their closest reference counterparts.

An *optimal control policy* is represented by

$$\text{IF } \overline{X}_t = \overline{S}_k \text{ THEN } \overline{U}_t = \overline{C}_l$$

$t = 1, \ldots, N-1$, $\overline{S}_k \in \overline{\mathscr{X}}$, $\overline{C}_l \in \overline{\mathscr{U}}$, equated with a fuzzy relation R_t^* in $X \times U$, and for a current X_t (not necessarily reference) U_t^* is determined by the *compositional rule of inference* $U_t^* = X_t \circ R_t^*$.

One can also use the earlier, simpler and more efficent branch-and-bound approach [, (Kacprzyk 1979)].

For lack of space we cannot discuss the other problem classes, i.e. those with an implicitly specified, fuzzy and infinite termination time. For most of them a dynamic programming problem can be formulated. For details we refer the reader to Kacprzyk's book [, (Kacprzyk 1997)].

7 Fuzzy multiobjective mathematical programming

In general, one can readily notice that in virtually all fuzzy mathematical programming models presented in the previous sections the objective functions (goals) and constraints have been treated alike. Therefore, from a certain point of view, those models could have been considered multiobjective mathematical programming models.

In this section we will briefly present models which are more specific to multiobjective fuzzy mathematical programming. Since there is a multitude of works in the literature on this topic, and our space is limited, we will state a general problem formulation, and then give a brief summary of main directions, approaches and solution techniques. We will follow in principle ([, Lai and Hwang 1992] and [, Lai and Hwang 1994]), and will outline the main problem classes.

The basic problem considered is of the type (30), i.e. for k objective functions and m constraints:

$$\begin{cases} & \widetilde{\max}_{x\in R}[f_1(c_1,x),\ldots,f_k(c_k,x)] \\ \text{subject to:} & g_j(a_j,x)\widetilde{\geq}0; j=1,\ldots,m \\ & x=[x_1,\ldots,x_n];x_i\geq 0, i=1,\ldots,n \end{cases} \tag{77}$$

where $\widetilde{\max}$ and $\widetilde{\geq}$ are meant that all the objective functions should be maximized, and this maximization is meant similarly as in (30), i.e. to attain possibly high a value, and the constraints should be satisfied as well as possible. The c_i, $i=1,\ldots,k$, and a_j, $j=1,\ldots,m$, are parameters which may be fuzzy or not.

As we may remember [cf. (35)], assuming piecewise linear membership functions, the solution of problem (77) may be obtained by solving an auxiliary problem ([, Zimmermann 1976], [, Zimmermann 1978]). Basically, first we calculate for each fuzzy objective and constraint (called for simplicity an objective, and denoted as the fuzzy objective), for $i=1,\ldots,k+m$:

- the individual best solution

$$f_i^* = \max_{x\in R} f_i(c_k,x) \tag{78}$$

- individual worst solution

$$f_i^- = \min_{x\in R} f_i(c_i,x) \tag{79}$$

- and assume the linear membership function of the i-th objective as

$$i(f_i) = \begin{cases} 1 & \text{for } f_i(c_i,x) > f_i^* \\ \frac{f_i(c_i,x)-f_i^-}{f_i^*-f_i^-} & \text{for } f_i^- \leq f_i(c_i,x) \leq f_i^* \\ 0 & \text{for } f_i(c_i,x) < f_i^- \end{cases} \tag{80}$$

Then, an optimal solution to problem (77) can be obtained by solving the following augmented maximin problem ([, Lai and Hwang 1992], [, Lai and Hwang 1994])

$$\begin{cases} & \max_{\lambda\in(0,1]} \lambda + \delta\sum_{i=1}^k w_k i(f_i) \\ \text{subject to:} & i(f_i)\geq\lambda; i=1,\ldots,k+m \\ & x\in R=[x_1,\ldots,x_n];x_i\geq 0, i=1,\ldots,n;\lambda\in(0,1] \end{cases} \tag{81}$$

where $\delta\in[0,1]$ is a sufficiently small number, and $w_i\in[0,1]$, $w_1+\cdots w_k=1$ are relative weights (importances) of the objective functions.

One can also set his or her goal f_i^0, $i=1,\ldots,k+m$, for the i-th fuzzy objective, and its corresponding tolerance t_i, such that $f_i^0\leq f_i^*$ and $f_i^*-t_i\geq f_i^-$, and then [, (Lai and Hwang 1992)] the augmented maximin problem, which is a counterpart of (81), is

$$\begin{cases} \max_{\lambda \in (0,1]} \lambda + \delta \sum_{i=1}^{k} w_k i(f_i) \\ \text{subject to: } i(f_i) = 1 - \frac{f_i^0 i f_i(c_i,x)}{t_i} \geq \lambda; i = 1,\dots,k+m \\ x \in R = [x_1,\dots,x_n]; x_i \geq 0, i = 1,\dots,n; \lambda \in (0,1] \end{cases} \tag{82}$$

Very often it is expedient to define a *parametric fuzzy multiobjective mathematical programming problem* because one may then perform some post-optimal analyses. Such a problem is given as

$$\begin{cases} \widetilde{\max}_{x \in R}[f_1(c_1,x),\dots,f_k(c_k,x)] \\ \text{subject to: } g_j(a_j,x) \tilde{\geq} b_j + \theta; j = 1,\dots,m \\ x \in R = [x_1,\dots,x_n]; x_i \geq 0, i = 1,\dots,n \end{cases} \tag{83}$$

where c_i and b_j may be fuzzy or not, and $\theta \in [0,1]$ is a variable parameter.

Then, we can follow the line of reasoning applied for prblem (77), i.e. (78)–(80), and abtain an augmented maximin problem which is equivalent to (81) and (82).

For more details on the above and some more specific problem classes we refer the reader to [, (Lai and Hwang 1994)].

Among a multitude of other approaches for solving various versions of multiobjective fuzzy optimization problems, one may also the following works. In [, (Zimmermann 1978)] an inherent incommensurability and conflicting nature of problem (77), which may be due to different characters of objective functions and an inability of their simultaneous maximization, is modeled by introducing some satisfaction levels on the objective functions and tolerances on the violation of constraints as in (32). Therefore, an equivalent to problem (35) is obtained which can be solved.

A fuzzy efficient solution was first defined in [, (Werners 1987)]. In ([, Hannan 1981a], [, Hannan 1981b]) the multiobjective fuzzy mathematical programming problem is solved by assuming fuzzy goals of a piecewise linear form. For a similar approach, see also [, (Ignizio 1982)]. This approach was later extended by numerous authors by including interaction with the decision maker (cf. [, (Tapia and Murtagh 1991)].

Sakawa and his collaborators were among the most active in developing various issues related to multiobjective fuzzy optimization as, e.g., concepts of Pareto optimality, compromise solutions, sequential proxy techniques, etc. These works are too numerous to cite, and the best source of information is Sakawa's book [, (Sakawa 1993)].

Among other approaches, in [, (Leung 1987)] a hierarchy of fuzzy objectives is employed, in [, (Chanas 1989)] a parametric model was proposed, etc.

For more information of different aspects of multiobjective fuzzy mathematical programming we refer the interested reader to the literature cited, the books [, (Lai and Hwang 1992)] and [, (Sakawa 1993)], and – maybe even better due to a more diversified coverage – to the edited volumes: [, (Delgado, Kacprzyk, Verdegay and Vila 1994)], [, (Fedrizzi, Kacprzyk and Roubens 1991)], and [, (Kacprzyk and Orlovski 1987)].

8 Concluding remarks

In this short review we have briefly presented foundations of fuzzy optimization and fuzzy mathematical programming. Emphasis was on linear fuzzy mathematical programming as the most promising for practical applications and most widely used. We have also presented some more important aspects of nonlinear fuzzy mathematical programming (in particular integer and 0–1 programming), fuzzy dynamic programming, and multiobjective fuzzy mathematical programming .

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