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Stability Analysis for Parametric Vector Optimization Problems

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Preface

We study stability of minimal points and solutions to parametric (or perturbed) vector optimization problems in the framework of real topological vector spaces and, if necessary, normed spaces. Because of particular importance of finite-dimensional problems, called multicriteria optimization problems, which model various real-life phenomena, a special attention is paid to the finite-dimensional case. Since one can hardly expect the sets of minimal points and solutions to be singletons, set-valued mappings are natural tools for our studies.

Vector optimization problems can be stated as follows. Let X be a topological space and let Y be a topological vector space ordered by a closed convex pointed cone $\mathcal{K} \subset Y$. Vector optimization problem

$$\begin{array}{ll} \mathcal{K} - \min\ f_0(x) & \\ \text{subject to } x \in A_0\,, & (P_0) \end{array}$$

where $f : X \to Y$ is a mapping, and $A_0 \subset X$ is a subset of X, relies on finding the set $\operatorname{Min}(f_0, A_0, \mathcal{K}) = \{y \in f_0(A_0) \mid f_0(A_0) \cap (y - \mathcal{K}) = \{y\}\}$ called the **Pareto** or **minimal point** set of (P_0), and the **solution set** $S(f_0, A_0, \mathcal{K}) = \{x \in A_0 \mid f_0(x) \in \operatorname{Min}(f_0, A_0, \mathcal{K})\}$. We often refer to problem (P_0) as the **original problem** or **unperturbed one**. The space X is the **argument** space and Y is the **outcome** space.

Let U be a topological space. We embed the problem (P_0) into a family (P_u) of vector optimization problems parametrised by a parameter $u \in U$,

$$\begin{array}{ll} \mathcal{K} - \min\ f(u,x) & \\ \text{subject to } x \in A(u)\,, & (P_u) \end{array}$$

where $f : U \times X \to Y$ is the parametrised objective function and $A : U \rightrightarrows Y$, is the feasible set multifunction, (P_0) corresponds to a parameter value u_0. The performance multifunction $\mathcal{M} : U \rightrightarrows Y$, is defined as $\mathcal{M}(u) = \text{Min}(f(u,\cdot), A(u), \mathcal{K})$, and the solution multifunction $\mathcal{S} : U \rightrightarrows Y$, is given as $\mathcal{S}(u) = S(f(u,\cdot), A(u), \mathcal{K})$, and $f : U \times X \to Y$, $A(u) \subset X$.

Our aim is to study continuity properties of $\mathcal{M}$ and $\mathcal{S}$ as functions of the parameter u. Continuous behaviour of solutions as functions of parameters is of crucial importance in many aspects of the theory of vector optimization as well as in applications(correct formulation of the model and/or approximation) and numerical solution of the problem in question.
We investigate continuity in the sense of Hausdorff and Hőlder of the multivalued mappings of minimal points $\mathcal{M}(u)$ and solutions $\mathcal{S}(u)$ as functions of the parameter u under possibly weak assumptions. We attempt to avoid as much as possible compactness assumptions which are frequently over-used (see eg [83]).
It is a specific feature of vector optimization that the outcome space is equipped with a partial order generated by a cone the properties of which are important for stability analysis. In many spaces cones of nonnegative elements have empty interiors and because of this we derive stability results for cones with possibly empty interior. This kind of results are specific for vector optimization and do not have their counterpart in scalar optimization.
We introduce two new concepts: the notion of containment(with some variants for cones with empty interiors),[16], and the notion of strict minimality, [12].

The containment property (CP), defined in topological vector spaces, is introduced to study upper semicontinuities (in the sense of Hausdorff) of minimal points, [11, 16]. It is a variant of the domination property (DP), which appears frequently in the context of stability of solutions to parametric vector optimization problems. Although it is not a commonly adopted view point, the domination property may be accepted as a solution concept which generalizes the standard concept of a solution to scalar optimization problem. In consequence, the containment property (CP) may also be seen as a solution concept in vector optimization. To investigate more deeply this aspect we interpret the containment property as a generalization of the concept of the set of $\phi-$local solutions appearing in the

context of Lipschitz continuity of solutions to scalar optimization problems. Under mild assumptions the containment property imply that the set weakly minimal points equals the set of minimal points. This equality, in turn, is a typical ingradient of standard finite-dimensional sufficient conditions for upper semicontinuity of minimal points.
To study Hőlder upper continuity of minimal points we define the rate of containment of a set with respect to a cone, which is a real-valued function of a scalar argument, see [14, 15]. The rate of growth of this function influence decisively the rate of Hőlder continuity of minimal points, [15].

Strictly minimal points are introduced to study lower semicontinuities (lower Hausdorff, lower Hőlder) of minimal points [20, 13]. The definition of a strictly minimal point is given in topological vector spaces and it is a generalization of the notion of a super efficient point in the sense of Borwein and Zhuang defined in normed spaces. We discuss strict minimality in vector optimization by proving that it is a vector counterpart of the concept of $\phi-$ local solution to scalar optimization problem.

Theory of vector optimization may be considered as an abstract study of optimization problems with mappings taking values in the outcome space equipped with a partial order structure. As such, it contains many concepts and results which generalize and/or have their counterparts in scalar optimization. The very definition of the set of minimal points of vector optimization problem in the outcome space may serve as an example here. This is a counterpart of the optimal value of scalar optimization problem. Another example is the concept of well-posed optimization problem. In subsequent developments we often compare our results and considerations with the corresponding approaches in scalar optimization. For instance, we define several classes of well-posed vector optimization problems by generalizing the concept of scalar minimizing sequence and in these classes we investigate continuity of solutions. For scalar optimization problems, the existing approaches and results on well-posedness are extensively discussed in the monograph by Dontchev and Zolezzi [33].

Convergence and rates of convergence of solutions to perturbed optimization problems is one of crucial topics of stability analysis in optimization both from theoretical and numerical points of view. For scalar optimization it was investigated by many authors see eg., [72], [32], [47], [78], [55], [81], [59], [60], [82], [2], and many others. An exhaustive survey of current state of research is given in the recent monograph by Bonnans and Shapiro [26]. In vector optimization the results on Lipschitz continuity of solutions are not so numerous, and concern some classes of problems, for linear case see eg.,[28], [29], [30], for convex case see eg., [25], [31].

The organization of the material is as follows. In Chapter 2 we investigate upper Hausdorff continuity of the multivalued mapping M, $M(u) = \mathrm{Min}(\Gamma(u)|\mathcal{K})$ assigning to a given parameter value u from a topological space U the set of minimal points of the set $\Gamma(u) \subset Y$ with respect to cone $\mathcal{K} \subset Y$, where for any subset A of a topological vector space Y the set of minimal points is defined as $\mathrm{Min}(A|\mathcal{K}) = \{y \in A \mid A \cap (y - \mathcal{K}) = \{y\}\}$, and $\Gamma : U \rightrightarrows Y$, is a given multivalued mapping. The main tool which allows us to obtain the general result is the containment property (CP). Some infinite-dimensional examples are discussed. A special attention is paid to the containment property (CP) in finite-dimensional case, when $Y = R^m$.

In Chapter 3 we discuss upper Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the rate of containment δ which is a one-variable nondecreasing function, defined for a given set A and the order generating cone $\mathcal{K}$. The assumption of sufficiently fast growth rate of this function appears to be the crucial assumption for all upper Hőlder stability results of Chapter 3.

In Chapter 4 we apply the results obtained in Chapters 2 and 3 to derive conditions for upper Hausdorff and upper Hőlder stability of minimal points to parametric vector optimization problems by taking $\Gamma(u) = f(u, A(u))$. Moreover, we introduce the concept of $\Phi-$ strong solutions to vector optimization problem (P_0), which is a generalization of the concept of a $\phi-$local minimizer to scalar optimization problem, the latter being introduced by Attouch and

Wets [6].

In Chapter 5 we investigate the lower continuity and lower Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the notion of strict minimality mentioned above and the rate of strict minimality. In Section 5.5 we apply the results obtained in Chapter 5 to parametric vector optimization problems and we derive sufficient conditions for lower and lower Hőlder continuity of Pareto point multivalued mapping $\mathcal{M}$. An important tool here is the notion of $\Phi-$ strict solution to vector optimization problem introduced in Section 6.1. This notion can be interpreted as another possible generalization of the concept of $\phi-$local minimizer.

In Chapter 6 we propose several definitions of a well-posed vector optimization problem. All these definitions are based on properties of $\varepsilon-$solutions to vector optimization problems. For well-posed vector optimization problems we prove upper Hausdorff continuity of solution multivalued mapping S, $S(u) = S(f(u,\cdot), A(u), \mathcal{K})$.

4

Upper Hausdorff continuity of minimal points to vector optimization problems

4.1 Φ–strong solutions to vector optimization problems.

In a series of publications Attouch and Wets [5],[6], [7] developed an approach to investigation of quantitative stability of variational systems as defined by Rockafellar and Wets [73]. These authors prove Lipschitz and Hölder continuity of solutions to scalar minimization problems under perturbations for ϕ–local minimizers. Given a function $f : X \to R$ an element $x_f \in X$ is called a ϕ–local minimizer of f if $f(y) \geq f(x_f) + \phi(\|y - x_f\|)$ for all y in some ball around x_f, with ϕ being an **admissible** function, ie. $\phi : R_+ \to R_+$, $\phi(t_n) \to 0$ implies $t_n \to 0$.

In this section we use similar approach to investigate stability of vector optimization problems.
Let $f : X \to Y$ be a mapping and $A \subset X$ be a subset of X. The vector optimization problem

$$\begin{array}{c} \mathcal{K} - min f(x) \\ \text{subject to } x \in A \end{array} \tag{22}$$

consists in finding all $x \in S(f, A, \mathcal{K}) = \{x \in A \mid f(x) \in Min(f(A)|\mathcal{K})\}$, $Min(f(A)|\mathcal{K}) = \{y \in f(A) \mid (y - \mathcal{K}) \cap f(A) = \{y\}\}$, (see Jahn [44], Luc [57]).

Definition 4.1.1 *The solution set $S(f, A, \mathcal{K})$ is called ϕ–**strong** or ϕ–**dominated** if for each $x \in A$ there exists $s_x \in S(f, A, \mathcal{K})$*

such that

$$f(x) \geq f(s_x)+\phi(\|x-s_x\|)\cdot B\,,\ ie.,\ f(x)-f(s_x)-\phi(\|x-s_x\|)\cdot B \in \mathcal{K}\,,$$

where $\phi : R_+ \to R_+$ *is a nondecreasing admissible function.*

Definition 4.1.2 *The solution set* $S(f, A, \mathcal{K})$ *is called* $\phi-$**strong** *or* $\phi-$**dominated** *of order* p *if for each* $x \in A$ *there exists* $s_x \in S(f, A, \mathcal{K})$ *such that*

$$f(x) \geq f(s_x)+c\|x-s_x\|^p\cdot B\,,\ ie.,\ f(x)-f(s_x)-\|x-s_x\|^p\cdot B \in \mathcal{K}\,.$$

Proposition 4.1.1 *Let* $\mathcal{K} \subset Y$ *be a closed convex cone in a normed space* Y, *and* $int\mathcal{K} \neq \emptyset$. *Let* $f : X \to Y$ *be a Lipschitz mapping defined on a normed space* X, *and let* $A \subset X$ *be a subset of* X. *If the solution set* $S(f, A, \mathcal{K})$ *is* $\phi-$*strong, then* (CP) *holds for* $f(A)$.

Proof. Suppose that the solution set $S(f, A, \mathcal{K})$ is $\phi-$strong. Because of the symmetry of balls in Y, for each $x \in A$ there exists $s_x \in S(f, A, \mathcal{K})$ such that

$$f(x) - f(s_x) + \phi(\|x - s_x\|) \cdot B \subset \mathcal{K}.$$

Since $\|f(x) - f(s_x)\| < L\|x - s_x\|$ and ϕ is nondecreasing $\phi(\frac{1}{L}\|f(x) - f(s_x)\|) \leq \phi(\|x - s_x\|)$ and

$$f(x)-f(s_x)+\phi(\frac{1}{L}\|f(x)-f(s_x)\|)\cdot B \subset f(x)-f(s_x)+\phi(\|x-s_x\|)\cdot B \subset \mathcal{K}.$$

Take $\varepsilon > 0$ and any $x \in A$ such that $\|f(x) - f(s_x)\| \geq d(f(x), Min(f(A)|\mathcal{K})) \geq L\varepsilon$. Hence,

$$f(x)-f(s_x)+\phi(\varepsilon)\cdot B \subset f(x)-f(s_x)+\phi(\frac{1}{L}\|f(x)-f(s_x)\|)\cdot B \subset \mathcal{K},$$

which , by Proposition 2.1.4, amounts to saying that (CP) holds for $f(A)$.

□

Proposition 4.1.2 *Let* $X = (X, \|\cdot\|)$ *and* $Y = (Y, \|\cdot\|)$ *be normed spaces. Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in* Y, *and* $int\mathcal{K} \neq \emptyset$, *and let* $A_0 \subset X$ *be a subset of* X.

Let $f : X \to Y$ be a Lipschitz mapping. If the solution set $S(f, A_0, \mathcal{K})$ is strong of order p, then the rate of containment of the set $f(A_0)$ satisfies

$$\delta(\varepsilon) \geq c\varepsilon^p .$$

Proof. Suppose that the solution set $S(f, A_0, \mathcal{K})$ is strong of order p. Because of the symmetry of balls in Y, for each $x \in A_0$ there exists $s_x \in S(f, A, \mathcal{K})$ such that

$$f(x) - f(s_x) + c\|x - s_x\|^p \cdot B \subset \mathcal{K} .$$

Since $\|f(x) - f(s_x)\| < L\|x - s_x\|$

$$\frac{c}{L^p}\|f(x) - f(s_x)\|^p \leq c\|x - s_x\|^p$$

and

$$f(x) - f(s_x) + \frac{c}{L^p}\|f(x) - f(s_x)\|^p \cdot B \subset f(x) - f(s_x) + c\|x - s_x\|^p \cdot B \subset \mathcal{K} .$$

Take $\varepsilon > 0$ and any $f(x) \in f(A_0)$ such that $d(f(x), Min(f(A_0)|\mathcal{K})) \geq L\varepsilon$. Clearly, $\|f(x) - f(s_x)\| \geq L\varepsilon$. Hence,

$$f(x) - f(s_x) + c\varepsilon^p \cdot B \subset f(x) - f(s_x) + \frac{c}{L^p}\|f(x) - f(s_x)\|^p \cdot B \subset \mathcal{K} ,$$

which means that $\delta(\varepsilon) \geq c\varepsilon^p$.

□

Let $f : X \to Y$, $A_0 \subset X$. Let $A : U \rightrightarrows Y$, be a set-valued mapping such that $A(u_0) = A_0$. Parametric vector optimization problem related to (22) has the form

$$\begin{array}{c} \mathcal{K} - min\ f(x) \\ \text{subject to } x \in A(u) \end{array} \tag{23}$$

The minimal point multifunction $\mathcal{M} : U \rightrightarrows Y$, is of the form

$$\mathcal{M}(u) = \{y \in f(A(u)) \mid (y - f(A(u))) \cap f(A(u)) = \{y\}\} ,$$

and $\mathcal{M}(u_0) = Min(f(A_0)|\mathcal{K})$. The solution set-valued mapping $S : U \rightrightarrows X$, takes the form

$$S(u) = \{x \in X \mid f(x) \in M(u)\} ,$$

and $S(u_0) = S(f, A_0, \mathcal{K})$.

Theorem 4.1.1 *Let $X = (X, \|\cdot\|)$, $Y = (Y, \|\cdot\|)$, $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $int\mathcal{K} \neq \emptyset$, $A_0 \subset X$ a subset of X, and let $f : X \to Y$ be a Lipschitz mapping defined on X, with constant L.*

Suppose that the solution set $S(f, A_0, \mathcal{K})$ of (22) is strong of order p with constant $c > 0$.

Suppose that one of the following conditions holds:

(i) *$Min(f(A_0)|\mathcal{K})$ is weakly compact,*

(ii) *$Min(f(A_0)|\mathcal{K})$ is bounded and weakly closed, and $\mathcal{K}$ has a weakly compact base.*

Then for any parametrization of the form (23) of the problem (22) such that

1. *A is upper Hőlder continuous at u_0 of order ℓ_1 with constant L_1,*
2. *A is lower Hőlder continuous at u_0 of order ℓ_2 with constant L_2,*

the minimal point set-valued mapping $\mathcal{M}$ is upper Hőlder continuous at u_0 of order $\min\{\ell_1, \frac{\min\{\ell_1,\ell_2\}}{p}\}$ with constant $\left(LL_1 + \frac{L^{\frac{1}{p}}(L_1+L_2)^{\frac{1}{p}}}{c}\right)$.

Proof. For the proof it is enough to observe that the set-valued mapping $f(A) : U \to Y$, which is the image of a upper (lower) Hőlder continuous set-valued mapping $A : U \to X$ under the Lipschitz mapping $f : X \to Y$ is upper (lower) Hőlder continuous.

Indeed, suppose that A is upper Hőlder continuous at u_0 of order ℓ_1 with constant L_1, ie there exists a neighbourhood U_0 of u_0 such that $A(u) \subset A(u_0) + L_1\|u - u_0\|^{\ell_1}$, ie for each $a \in A(u)$ there exists $a_0 \in A(u_0$ such that $\|a - a_0\| < L_1\|u - u_0\|^{\ell_1}$. Since the mapping f is Lipschitzian with constant L

$$1/L\|f(a) - f(a_0)\| < \|a - a_0\| < L_1\|u - u_0\|^{\ell_1},$$

which proves that $f(A)$ is upper Hőlder continuous at u_0 of order ℓ_1 with constant LL_1. The proof of lower Hőlder continuity of $f(A)$ is analogous. Now the result follows from Theorem 3.2.1 and Proposition 4.1.2.

□

Definition 4.1.3 *A function $f : X \to Y$ is called $\phi-$ convex if*

$$f(tx_1 + (1-t)x_2) \in tf(x_1) + (1-t)f(x_2) + \phi(\|x_2 - x_1\|)B - \mathcal{K}$$

for all $x_1, x_2 \in X$, $0 \leq t \leq 1$.

A function f which is $\phi-$convex with $\phi(r) = C \cdot r^\gamma$, $1 < \gamma \leq 2$is called $\mathcal{K} - \gamma-$paraconvex. When $\gamma = 2$, a $\mathcal{K} - 2-$paraconvex function is called $\mathcal{K}-$uniformly convex.
In a series of papers ([74],[75],[76]), Rolewicz investigated $\gamma-$paraconvex functions and multifunctions. A multifunction $\Gamma : X \to Y$ is called $\gamma-$paraconvex, where $1 < \gamma \leq 2$, if there exists a constant $C > 0$ such that

$$\lambda\Gamma(x_1) + (1-\lambda)\Gamma(x_2) \subset \Gamma(\lambda x_1 + (1-\lambda)x_2) + C\|x_1 - x_2\|^\gamma \cdot B.$$

Let us define a set-valued mapping $\Gamma : X \to Y$ as $\Gamma(x) = f(x) + \mathcal{K}$.

Proposition 4.1.3 *A function $f : X \to Y$ is $\mathcal{K} - \gamma-$paraconvex if and only if a multifunction $\Gamma : X \to Y$, $\Gamma(x) = f(x) + \mathcal{K}$ is $\gamma-$paraconvex.*

Proof. Let f be $\mathcal{K} - \gamma-$paraconvex and let $x_1, x_2 \in \mathrm{dom}\Gamma$. Take any $y_1 \in \Gamma(x_1)$, $y_2 \in \Gamma(x_1)$. We have $y_1 = f(x_1) + k_1$, $y_2 = f(x_2) + k_2$, and by $\mathcal{K} - \gamma-$paraconvexity of f,

$$f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda)f(x_2) + C\|x_1 - x_2\|^\gamma \cdot b_\lambda - k_\lambda,$$

ie.

$$\lambda f(x_1) + (1-\lambda)f(x_2) \in \Gamma(\lambda x_1 + (1-\lambda)x_2) + C\|x_1 - x_2\|^\gamma \cdot B + \mathcal{K}.$$

On the other hand, suppose that Γ is $\gamma-$paraconvex. Hence, there exist $k \in \mathcal{K}$, $b \in B$ such that

$$\lambda f(x_1) + (1-\lambda)f(x_2) = f(\lambda x_1 + (1-\lambda)x_2) + k + C\|x_1 - x_2\|^\gamma \cdot b,$$

and consequently

$$f(\lambda x_1+(1-\lambda)x_2)=\lambda f(x_1)+(1-\lambda)f(x_2)-k-C\|x_1-x_2\|^{\gamma}\cdot b,$$

which means that f is $\mathcal{K}-\gamma-$paraconvex.

□

Let us recall that a closed convex subset $A\subset X$ of a linear normed space X is **uniformly convex** (see eg. [42]) if there exists a nondecreasing function κ on $[0,+\infty)$ with $0=\kappa(0)<\kappa(r)$, $r>0$, such that

$$\frac{1}{2}(x+y)+z\in A$$

whenever $x,y\in A$ and $\|z\|\leq\kappa(\|x-y\|)$.
A closed convex subset $A\subset Y$ is **uniformly convex of order p** if there exists a nondecreasing function κ on $[0,+\infty)$ with $0=\kappa(0)<\kappa(r)$, $r>0$, such that

$$\frac{1}{2}(x+y)+z\in A$$

whenever $x,y\in A$ and $\|z\|\leq\kappa(\|x-y\|^{p})$,

Proposition 4.1.4 *Let $A\subset X$ be a uniformly convex subset of the space X, and let $f:X\to Y$ be a $\mathcal{K}-\gamma-$paraconvex function. Then the containment property holds for $f(A)$, and the solution set is strong.*

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