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ESSAYS ON STABILITY ANALYSIS AND MODEL REDUCTION

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Chapter 2

Properties of Routh's reduction method

The Routh algorithm was conceived to analyze the stability of dynamic systems but can be applied in different contexts [1], such as model reduction and the computation of certain performance indexes (cf., e.g., [2] and [3]). In particular, it can be used to compute the integral along the imaginary axis of the square magnitude of a stable rational function of a complex variable, which is related to the "energy" of its inverse Laplace transform [4]. More generally, the Routh array can be used to compute the entries of the impulse–response Gramian of a dynamic system, whose diagonal entries are the energies of the impulse response and of some of its derivatives and whose off–diagonal entries can be obtained by adding to such energies other terms that depend on the Markov parameters, i.e., the coefficients of the asymptotic expansion of the system transfer matrix [7].

This chapter focuses on the energy–retention property of the so–called Routh approximation [4]. Precisely, it is shown that the all–pole transfer function $1/P_i(s)$ whose denominator $P_i(s)$ is formed from two consecutive rows of the Routh array for a given Hurwitz polynomial $P_n(s)$, $n > i$, matches a suitable number of impulse–response energies of the system with transfer function $1/P_n(s)$. This property can be exploited to find reduced–order models that retain some energies of an original high–order system. It is also shown that these energies can easily be evaluated from the entries of the Routh table for $P_n(s)$. Finally, the extension to the case of transfer functions with zeros is briefly outlined.

2.1 Two–term recursion

As shown in Chapter 1, the standard Routh algorithm generates a sequence of polynomials of descending degree starting from the even and odd parts $Q_n(s)$ and $Q_{n-1}(s)$ if n is even, or vice versa if n is odd, of a given polynomial

$$P_n(s) = Q_n(s) + Q_{n-1}(s), \tag{2.1}$$

according to the recursion:

$$Q_{i-2}(s) = Q_i(s) - q_{i-1}\, s\, Q_{i-1}(s), \tag{2.2}$$

where q_{i-1} is the ratio of the leading coefficients of $Q_i(s)$ and $Q_{i-1}(s)$, respectively. The entries of every row of the Routh table are precisely the coefficients of the decreasing powers of s in the polynomials:

$$Q_i(s) = \sum_{k=0}^{\lfloor i/2 \rfloor} r_{i,i-2k} s^{i-2k}, \;\; 0 \leq i \leq n. \tag{2.3}$$

Using this notation, the polynomial (2.1) can be written as

$$P_n(s) = \sum_{k=0}^{\lfloor n/2 \rfloor} r_{n,n-2k} s^{n-2k} + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} r_{n-1,n-1-2k} s^{n-1-2k} \tag{2.4}$$

and q_{i-1} in (2.2) as

$$q_{i-1} = \frac{r_{i,i}}{r_{i-1,i-1}}, \;\; 1 \leq i \leq n. \tag{2.5}$$

Assuming that all of the leading coefficients $r_{i,i}$ of $Q_i(s)$ are different from zero, which is certainly true when $P_n(s)$ is a Hurwitz polynomial, a *complete* polynomial

$$P_i(s) = Q_i(s) + Q_{i-1}(s), \;\; 1 \leq i \leq n-1, \tag{2.6}$$

can also be associated with every pair of consecutive polynomials $Q_i(s)$ in the sequence generated by (2.2) from $Q_n(s)$ and $Q_{n-1}(s)$. In this way, the sequence of $n+1$ polynomials:

$$\{P_i(s), i = 0, \cdots, n\} \tag{2.7}$$

can be formed. Clearly, two consecutive polynomials in this sequence have either the even or the odd part in common. Note that all of the polynomials in (2.7) are Hurwitz because their Routh table coincides with the lower part of the table for the Hurwitz polynomial $P_n(s)$ from the row of order i to that of order 0.

By combining (2.6) and (2.2), it turns out that two consecutive complete polynomials in (2.7) are related by means of the step–down (or backward) recursion:

$$P_{i-1}(s) = \left(1 + \frac{q_{i-1}}{2}s\right)P_i(s) - (-1)^i \frac{q_{i-1}}{2} sP_i(-s), \;\; i \le n, \tag{2.8}$$

which is called the two–term form of Routh's algorithm [4] as opposed to the three–term form (2.2). Conversely, it is possible to obtain a polynomial of immediately higher degree from any polynomial in the sequence according to the two–term step–up (or forward) recursion:

$$P_i(s) = \left(1 - \frac{q_{i-1}}{2}s\right)P_{i-1}(s) + (-1)^i \frac{q_{i-1}}{2} sP_{i-1}(-s), \;\; i \le n. \tag{2.9}$$

2.2 Energy retention

The Routh approximation method (see, e.g., [2]) generates a sequence of *stable* reduced–order transfer functions approximating an original stable nth–order transfer function

$$G_n(s) = \frac{1}{P_n(s)} \tag{2.10}$$

according to

$$G_i(s) = \frac{1}{P_i(s)}, \;\; i < n, \tag{2.11}$$

where the denominators $P_i(s)$ are obtained recursively from the Hurwitz denominator of (2.10) by means of (2.8). It is shown next that, besides stability, the models (2.11) retain a number of impulse–response energies of the original model.

Denoting by $g_i^{(h)}(t)$, $0 \le h < i$, the hth derivative of the impulse response of $G_i(s)$, $i \le n$, so that, $g_i^{(0)}(t) = \mathrm{LT}^{-1}[G_i(s)]$, the energy $J_{i,h}$ of $g_i^{(h)}(t)$ is defined as

$$J_{i,h} := \int_0^\infty [g_i^{(h)}(t)]^2 dt. \tag{2.12}$$

By Parseval's theorem

$$J_{i,h} = \frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{s^h(-s)^h}{P_i(s)P_i(-s)}\, ds\,, \quad h < i. \tag{2.13}$$

According to (2.8), at the LHP roots of the Hurwitz polynomial $P_i(s)$

$$P_{i-1}(-s) = \left(1 - \frac{q_{i-1}}{2}s\right) P_i(-s) \tag{2.14}$$

and, therefore, by the residue theorem

$$J_{i,h} = \frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{s^h(-s)^h\left(1 - \frac{q_{i-1}}{2}s\right)}{P_i(s)P_{i-1}(-s)}\, ds\,, \quad h < i-1. \tag{2.15}$$

Similarly, according to (2.9), at the RHP roots of $P_{i-1}(-s)$

$$P_i(s) = \left(1 - \frac{q_{i-1}}{2}s\right) P_{i-1}(-s) \tag{2.16}$$

and, therefore,

$$J_{i,h} = \frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{s^h(-s)^h}{P_{i-1}(s)P_{i-1}(-s)}\, ds\,, \quad h < i-1. \tag{2.17}$$

It follows that

$$J_{i-1,h} = J_{i,h} =: J_h, \quad h < i-1. \tag{2.18}$$

This result can be stated as follows.

Theorem 2.2.1 *The system with transfer function (2.11) with $P_i(s)$ obtained from $P_n(s)$ according to the Routh algorithm (2.8) retains the first i energies $J_{n,h}$, $h = 0, \cdots, i-1$, of the system with transfer function (2.10).* □

2.3 Evaluation of the energies

Using (2.3), any polynomial $P_i(s)$ in the sequence (2.7) can be expressed as

$$P_i(s) = \sum_{k=0}^{\lfloor i/2 \rfloor} r_{i,i-2k} s^{i-2k} + \sum_{k=0}^{\lfloor (i-1)/2 \rfloor} r_{i-1,i-1-2k} s^{i-1-2k}\,, \tag{2.19}$$

where the coefficients $r_{i,i-2k}$ and $r_{i-1,i-1-2k}$ are the entries of the row of order i and, respectively, $i-1$ in the Routh table for $P_n(s)$.
For $t > 0$, the impulse response $g_i(t)$ of the system with transfer funtion (2.11) satisfies the homogeneous equation:

$$\sum_{k=0}^{\lfloor i/2 \rfloor} r_{i,i-2k}\, g_i^{(i-2k)}(t) + \sum_{k=0}^{\lfloor (i-1)/2 \rfloor} r_{i-1,i-1-2k}\, g_i^{(i-1-2k)}(t) = 0\,. \tag{2.20}$$

Multiplying (2.20) by $g_i^{(i-2)}(t)$ and integrating from 0 to ∞, we obtain:

$$\sum_{k=0}^{\lfloor i/2 \rfloor} r_{i,i-2k} \int_0^\infty g_i^{(i-2k)}(t) g_i^{(i-2)}(t)\, dt \; + \\ \sum_{k=0}^{\lfloor (i-1)/2 \rfloor} r_{i-1,i-1-2k} \int_0^\infty g_i^{(i-1-2k)}(t) g_i^{(i-2)}(t)\, dt = 0\,. \tag{2.21}$$

By successively integrating by parts and taking into account (2.18), we find:

$$\int_0^\infty g_i^{(j)}(t) g_i^{(i-2)}(t)\, dt = 0\,, \quad i-j \text{ odd}\,, \tag{2.22}$$

and

$$\int_0^\infty g_i^{(j)}(t) g_i^{(i-2)}(t)\, dt = (-1)^{\frac{i-j}{2}-1} J_{\frac{i+j}{2}-1}\,, \quad i-j \text{ even}\,, \tag{2.23}$$

so that (2.21) becomes

$$\sum_{k=0}^{\lfloor i/2 \rfloor} r_{i,i-2k}\, J_{i-k-1} = 0\,. \tag{2.24}$$

Since $P_n(s)$ is Hurwitz, $r_{i,i} \neq 0$ and (2.24) can be solved for J_{i-1} leading to

$$J_{i-1} = \frac{1}{r_{i,i}} \sum_{k=1}^{\lfloor i/2 \rfloor} r_{i,i-2k}\, J_{i-k-1}\,. \tag{2.25}$$

It follows that the energies J_h, $h \leq n-1$, can be computed recursively from the entries of the rows of the Routh table for $P_n(s)$ starting from

$$J_0 = J_{1,0} = \frac{1}{2 r_{1,1} r_{0,0}}\,, \tag{2.26}$$

where $r_{1,1}$ and $r_{0,0}$ are the unique entries of the rows of order 1 and 0, respectively.

2.4 Extension to general transfer functions

Consider the stable nth–order transfer function:

$$F_n(s) = \frac{N_m(s)}{P_n(s)}\,, \tag{2.27}$$

where

$$N_m(s) = \sum_{k=0}^{m} b_{m,k}\, s^k\,,\ \ m < n, \tag{2.28}$$

with $N_m(s)$ and $P_n(s)$ coprime and $P_n(s)$ monic, and denote by $f_n(t)$ the inverse Laplace transform of $F_n(s)$.

Using a notation consistent with the one in Section 2.2, the impulse–response energies for the system characterized by (2.27) are

$$I_h := \int_0^{\infty} [f_n^{(h)}(t)]^2\, dt = \frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} s^h F(s) F(-s)(-s)^h ds,\ \ 0 \leq h < n-m. \tag{2.29}$$

For $s = \jmath\omega$, the numerator of the right–hand integrand in (2.29) can be written as

$$\omega^{2h} N_m(\jmath\omega) N_m(-\jmath\omega) = \omega^{2h} |N_m(\jmath\omega)|^2 = \sum_{k=0}^{m} B_{2k} \omega^{2(k+h)}, \tag{2.30}$$

where

$$B_{2k} = b_{m,k}^2 + 2 \sum_{j=1}^{k} (-1)^j b_{m,k-j} b_{m,k+j} \tag{2.31}$$

with $b_{m,k} = 0$ for $k > m$. It follows that

$$I_h = \sum_{k=0}^{m} B_{2k} J_{k+h}\,, \quad h < n-m. \tag{2.32}$$

In conclusion, the procedure to compute I_h entails the following steps:
(i) form the Routh table for $P_n(s)$;
(ii) starting from (2.26), compute $J_1, J_2, \cdots, J_{m+h}$ by means of (2.25);
(iii) evaluate the coefficients B_{2k} according to (2.31);
(iv) compute I_h using (2.32).

2.5 Computational complexity

The number of computations needed to compute I_0 depends on the numerator and denominator degrees n and m of (2.27) as follows. Let ν_a, ν_m and ν_d be the number of additions, multiplications and divisions, respectively. The numbers of elementary algebraic operations required by Step (i) of the procedure at the end of Section 2.4 are: $\nu_a = [(n-2)n]/4$ for n even and $\nu_a = (n-1)^2/4$ for n odd, $\nu_m = \nu_a$, $\nu_d = n-2$. At Step (ii): $\nu_a = (m^2-4m+4)/4$ for m even and $\nu_a = (m^2-4m+3)/4$ for m odd, $\nu_m = (m^2+4)/4$ for m even and $\nu_m = (m^2+3)/4$ for m odd, $\nu_d = m+1$. At Step (iii): $\nu_a = m^2/2$ for m even and $\nu_a = (m^2-1)/2$ for m odd, $\nu_m = (m^2+4m+4)/4$ for m even and $\nu_m = (m^2+4m+3)/4$ for m odd, $\nu_d = 0$. At Step (iv): $\nu_a = m$, $\nu_m = m+1$, $\nu_d = 0$. Therefore, the total numbers of operations are:

$$\nu_a = \frac{1}{4}\left(n^2 - 2n + 3m^2 + k_a\right), \tag{2.33}$$

$$\nu_m = \frac{1}{4}\left(n^2 - 2n + 2m^2 + 8m + k_m\right), \tag{2.34}$$

$$\nu_d = n + m - 1\,, \tag{2.35}$$

where
- $k_a = 4$ for n and m even, $k_a = 5$ for n odd and m even, $k_a = 1$ for n even and m odd, $k_a = 2$ for n and m odd,
- $k_m = 12$ for n and m even, $k_m = 13$ for n odd and m even, $k_m = 10$ for n even and m odd, $k_m = 11$ for n and m odd.

Note, by way of comparison, that the method suggested in [5] to compute I_0 requires the following numbers of operations:

$$\nu_a = \frac{1}{4}\left(2n^2 + 2n + 4\right), \tag{2.36}$$

$$\nu_m = \frac{1}{4}\left(2n^2 - 2n + 4\right), \tag{2.37}$$

$$\nu_d = 3n - 5\,. \tag{2.38}$$

Consequently, the method presented in [4] is preferable to that in [5] for n large. Further advantages arise when some indexes I_h, $h > 0$, need be computed or when the energy indexes for a number of rational functions with the same denominator are to be considered simultaneously, as is often the case in MIMO systems.

2.6 Concluding remarks

It has been shown that Routh's reduction method not only ensures the stability of the approximant but also leads to the retention of some energy indexes, which explains why it usually gives rise to satisfactory models. From this point of view, the simplification methods in [6], [7] and [3] can be considered as the natural extensions of Routh's approximation.

It has been pointed out that each energy J_h (see (2.12)), $h < n$, associated with the all–pole transfer function (2.11) is a linear combination of some energies J_l, $l < h$, according to the entries of the row of order $h + 1$ in the Routh table for the denominator of (2.11). On this basis, a recursive procedure to evaluate the quadratic integrals (2.29) associated with the transfer function (2.27) has been suggested, which represents an interesting alternative to the method presented in [5].

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