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Chapter 3

Common setting of stability tests

Stability–test algorithms are applied in many contexts, such as stability analysis, model reduction, signal processing, filter synthesis and robust control. Therefore, many authors have carefully studied and compared their properties in both the s–domain (continuous–time systems) and the z–domain (discrete–time systems) [1] $\div$ [18].

A unified interpretation of the available algorithms provides a new insight into their operation [13], [14] and may help find new procedures with specific characteristics. This chapter shows that all of the recursive stability–test algorithms can be given a common setting that is suggestive of new algorithms. Attention is limited to continuous–time systems, but similar considerations could be developed for discrete–time systems. It turns out that the aforementioned procedures can conveniently be classified according to the configuration of the root loci into which their basic recursions can be embedded.

3.1 General two–term recursions

A number of recursive stability–test algorithms have been proposed in the literature. Concerning continuous-time systems, besides the classical Routh test, mention can be made of the Lepschy test [9], [13] and the Euclid–type procedure suggested in [9]. All of these algorithms can be expressed in (at least) two different forms that, excluding the critical cases, allow us to generate recursively a sequence $\{P_i(s), i = n, \cdots, 0\}$

of $n+1$ polynomials $P_i(s)$ of descending degree i starting from an original polynomial $P_n(s)$ of degree n. This section concentrates on their *two–term* recursive form, by which every complete polynomial $P_{i-1}(s)$ currently generated by their basic step–down recursion is related to the complete polynomial $P_i(s)$ of immediately higher degree. The alternative *three–term* (or *split* [2], [3]) form, which relates either the even or the odd parts of three consecutive complete polynomials, is analyzed in [8]

The two–term step–down form of Routh's algorithm, already considered in Chapter 2, is:

$$P_{i-1}(s) = \Big(1 + \frac{q_{i-1}}{2}s\Big)P_i(s) - (-1)^i \frac{q_{i-1}}{2} s P_i(-s). \tag{3.1}$$

The basic recursion of Lepschy's test [13] is instead:

$$(s+1)P_{i-1}(s) = h_i \Big[P_i(s) + q_{L,i-1}P_i(-s)\Big] \tag{3.2}$$

with

$$h_i = \frac{1}{1 + q_{L,i-1}} \tag{3.3}$$

and that for the Euclid–type test [9] is:

$$(s^2-1)P_{i-1}(s) = \frac{1}{2}\Big[s(s - q_{E,i-1})P_i(s) - (-1)^i(s^2 - q_{E,i-1}s - 2)P_i(-s)\Big] \tag{3.4}$$

which, like the Routh algorithm (2.2), admits a much simpler three–term form. As is known, the location of the n roots of the original polynomial $P_n(s)$ can be related to the values taken by the n parameters q_i, $q_{L,i}$ or $q_{E,i}$, $i = n-1, n-2, \cdots, 1$, at every step of the above procedures.

In (3.1), (3.2) and (3.4), the polynomial $P_{i-1}(s)$ of lower degree is obtained from a suitable combination of the polynomials $P_i(s)$ and $P_i(-s)$ whose roots are symmetric with respect to the imaginary axis. To achieve this result, either some coefficients in the highest powers of s in this combination are forced to be zero as in (3.1), or this combination is forced to have $s+1$ or s^2-1 as a factor (see (3.2) and (3.4), respectively). It follows that the previous recurrence relations are particular cases of the more general recursion:

$$R_i(s)P_{i-1}(s) = F_i(s)P_i(s) + G_i(s)P_i(-s), \tag{3.5}$$

where the polynomials $R_i(s)$, $F_i(s)$ and $G_i(s)$ must satisfy the following conditions:
(i) the roots of $R_i(s)$ are roots of the polynomial $F_i(s)P_i(s)+G_i(s)P_i(-s)$,
(ii) $P_i(s)$ is Hurwitz if and only if $P_{i-1}(s)$ is Hurwitz and the coefficients of $F_i(s)$ and $G_i(s)$ belong to suitable domains that depend on the choice of $R_i(s)$.

Once $R_i(s)$ has been chosen, the first condition can easily be expressed in analytic terms leading to a set of equations linear in the coefficients of $P_{i-1}(s)$, $F_i(s)$ and $G_i(s)$. It is more difficult to translate the second condition into constraints on the remaining free parameters. In any case, conditions (i) and (ii) do not uniquely determine all of the available parameters, so that different families of stability–test algorithms may be generated.

In the following, attention is focused on the simplest form of (3.5) capable of generating both the Routh and the Lepschy algorithm, that is,

$$(r_{i,0} + r_{i,1}s)P_{i-1}(s) = (f_{i,0} + f_{i,1}s)P_i(s) + (g_{i,0} + g_{i,1}s)P_i(-s) \tag{3.6}$$

which reduces to (3.1) for $r_{i,0} = 1$, $r_{i,1} = 0$, $f_{i,0} = 1$, $f_{i,1} = q_{i-1}/2$, $g_{i,0} = 0$, $g_{i,1} = -(-1)^i q_{i-1}/2 = -(-1)^i f_{i,1}$ and to (3.2) for $r_{i,0} = 1$, $r_{i,1} = 1$, $f_{i,0} = 1/[1 + q_{L,i}]$, $f_{i,1} = 0$, $g_{i,0} = q_{L,i}/[1 + q_{L,i}] = q_{L,i}f_{i,0}$, $g_{i,1} = 0$. Since $r_{i,0} = 0$ cannot lead to a stability test and the results obtainable for $f_{i,0} = 0$ correspond to those for $g_{i,0} = 0$ by replacing $P_i(s)$ with $P_i(-s)$, the analysis can be restricted to the relation:

$$(1 + r_{i,1}s)P_{i-1}(s) = (1 + f_{i,1}s)P_i(s) + (g_{i,0} + g_{i,1}s)P_i(-s), \tag{3.7}$$

where $r_{i,0} = f_{i,0} = 1$, thus excluding from consideration the polynomials that differ from $P_i(s)$ and $P_{i-1}(s)$ by a real proportionality factor.

According to condition (i) above, $P_{i-1}(s)$ is indeed of degree $i - 1$ only if

$$g_{i,1} = -(-1)^i f_{i,1}, \tag{3.8}$$

and the right–hand side of (3.7) admits a root in $1/r_{i,0}$ only if

$$g_{i,0} = k_i - \frac{f_{i,1}}{r_{i,1}}[k_i + (-1)^i] \tag{3.9}$$

with

$$k_i = -P_i(-1/r_{i,1})/P_i(1/r_{i,1}). \tag{3.10}$$

The next section shows that (3.7) gives rise essentially to three families of stability–test algorithms that differ from one another by the configuration of the root loci into which their basic recursions can be embedded.

3.2 Lepschy–like algorithms

Taking (3.8), (3.9) and (3.10) into account, for $f_{i,1} = 0$ relation (3.7) becomes

$$(1 + r_{i,1}s)P_{i-1}(s) = P_i(s) + k_i P_i(-s) \tag{3.11}$$

which includes (3.2) as a particular case (except for the normalization factor h_i).

The roots of the left–hand side of (3.11) belong to the complete root locus ($k \in \mathbb{R}$) of the equation:

$$P_i(s) + kP_i(-s) = 0 \tag{3.12}$$

for $k = k_i$. Assuming $P_i(s)$ and $P_i(-s)$ coprime, this locus is a real algebraic variety consisting of i branches on which k is a coordinate. These branches "depart" from the i roots of $P_i(s)$ for $k = 0$ and "arrive" at the i opposite roots of $P_i(-s)$ as $k \to \pm\infty$. Therefore, *the locus is symmetric with respect to the imaginary axis* and the intersection of its branches with the same axis, if any, occur for $|k| = 1$.

When $P_i(s)$ is a Hurwitz polynomial, all of the roots of (3.12) are in the LHP for $|k| < 1$. If $r_{i,1} > 0$ and, thus, the root of $1 + r_{i,1}s$ lies in the LHP, the corresponding value k_i of the current parameter k is in the open interval $(-1, +1)$ and all of the roots of (3.12) are in the LHP. A typical root locus for a Hurwitz polynomial $P_i(s)$ is shown in Fig. 3.1.

On the basis of these considerations, a family of stability tests can be derived depending on the value of $r_{i,1}$. For a proof of the Lepschy stability test, which corresponds to $r_{i,1} = 1$, the interested reader is referred to [13]. It is straightforward to extend the proof to the case of arbitrary $r_{i,1} > 0$.

3.3 Routh–like algorithms

A different family of algorithms that includes the Routh test is obtained by setting $g_{i,0} = 0$ in (3.7). Precisely, the classical Routh algorithm

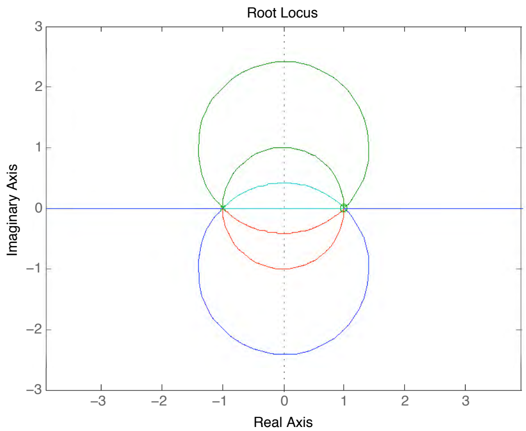

Figure 3.1: Root locus for (3.12) with $P_i(s) = P_4(s) = (s+1)^4$.

corresponds to the additional constraint $r_{i,1} = 0$, but similar properties hold for arbitrary $r_{i,1} < 0$.

The recurrence relations characterizing all of these algorithms can be embedded in an *asymmetric root locus* whose branches never cross the imaginary axis except for one branch laying over the real axis (at least partly). Under the present assumptions and taking (3.8) into account, (3.7) particularizes to

$$(1 + r_{i,1}s)P_{i-1}(s) = P_i(s) + 2f_{i,1}sQ_{i,i-1}(s), \tag{3.13}$$

where

$$Q_{i,i-1}(s) = \frac{1}{2}[P_i(s) - (-1)^i P_i(-s)] \tag{3.14}$$

is the odd part of $P_i(s)$ if i is even and its even part otherwise.

The roots of (3.13) belong to the complete root locus ($k \in \mathbb{R}$) of the equation:

$$P_i(s) + ksQ_{i,i-1}(s) = 0 \tag{3.15}$$

for

$$k = 2f_{i,1} = r_{i,1}\frac{P_i(-1/r_{i,1})}{Q_{i,i-1}(-1/r_{i,1})}. \tag{3.16}$$

The departure and arrival points of this locus are: (i) the roots of $P_i(s)$ for $k = 0$, and (ii) the roots of $sQ_{i,i-1}(s)$ for $k = \pm\infty$. Following a procedure similar to that adopted in Section 1.2 of Chapter 1, it can be shown that, excluding the case in which $P_i(s)$ and $sQ_{i,i-1}(s)$ have common factors, $i-1$ locus branches never touch the imaginary axis for a finite value of k (and, thus, they never cross it) and the remaining branch passes through the point at infinity going from the right half of the real axis to the left half of the real axis, or vice versa, for the (finite) value of k that lowers the degree of (3.15) by one. Denoting again $P_i(s)$ by

$$P_i(s) = \sum_{j=0}^{i} a_{i,k}\, s^k, \tag{3.17}$$

this value is

$$k = -\frac{a_{i,i}}{a_{i,i-1}}. \tag{3.18}$$

A typical root locus for a Hurwitz polynomial $P_i(s)$ is shown in Fig. 3.2.

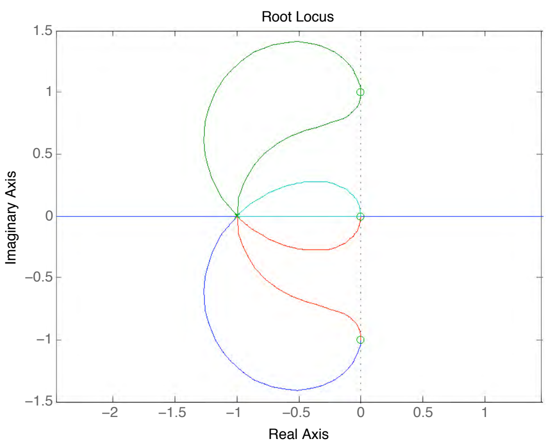

Figure 3.2: Root locus for (3.15) with $P_i(s) = P_4(s) = (s+1)^4$.

On the basis of the previous considerations, the following theorem can be proved along lines similar to those followed in Section 1.2 of Chapter 1 (see [17]).

Theorem 3.3.1 *The polynomial $P_i(s)$ is Hurwitz and $r_{i,1} \leq 0$ if and only if $P_{i-1}(s)$ is Hurwitz and*

$$0 \geq r_{i,1} > r_{i,c} := -\frac{a_{i-1,i-1}}{a_{i-1,i-2}}, \tag{3.19}$$

$$f_{i,1} < f_{i,c} := -\frac{1}{2}\frac{r_{i,1}r_{i,c}}{r_{i,1} - r_{i,c}}. \tag{3.20}$$

□

From the above proposition, a stability criterion generalizing the standard Routh criterion can be obtained easily. In fact, starting from the nth–degree polynomial $P_n(s)$ to be tested, a sequence of polynomials $P_i(s)$, $i = n, n-1, \cdots, 1, 0$, is formed according to (3.13) with $r_{i,1} \leq 1$. If (3.19) and (3.20) are satisfied at every step, then $P_n(s)$ is Hurwitz. The algorithm can also be used to evaluate the numbers of LHP and RHP roots of a polynomial [17].

3.4 Mixed–type algorithms

Relation (3.7) gives rise to yet another interesting family of stability–test procedures that have been called *mixed–type algorithms* [17] because they exhibit intermediate characteristics between those of the Routh–like and Lepschy–like algorithms. To show this, consider, besides constraints (3.8) and (3.9), the further constraint:

$$g_{i,0} = -g_{i,1} = (-1)^i f_{i,1}. \tag{3.21}$$

Then, recursion (3.7) takes the form:

$$(1 + r_{i,1}s)P_{i-1}(s) = (1 + f_{i,1}s)P_i(s) + (-1)^i f_{i,1}(1-s)P_i(-s), \tag{3.22}$$

which may be rewritten as

$$\frac{1}{1-f_{i,1}}(1+r_{i,1}s)P_{i-1}(s) = P_i(s) + \frac{2f_{i,1}}{1-f_{i,1}}[Q_{i,i}(s) + sQ_{i,i-1}(s)], \tag{3.23}$$

where $Q_{i,i-1}(s)$ is given by (3.14),

$$Q_{i,i}(s) = \frac{1}{2}[P_i(s) + (-1)^i P_i(-s)] \tag{3.24}$$

and, according to (3.9) and (3.10),

$$f_{i,1} = \frac{k_i r_{i,1}}{k_i + (-1)^i(1 + r_{i,1})}. \tag{3.25}$$

In this case, the roots of (3.22) belong to the root locus of

$$P_i(s) + k[Q_{i,i}(s) + sQ_{i,i-1}(s)] = 0 \tag{3.26}$$

for

$$k = \frac{2f_{i,1}}{1 - f_{i,1}}. \tag{3.27}$$

Excluding again the case of common roots between $Q_{i,i}(s)$ and $Q_{i,i-1}(s)$, the locus consists of i branches on which k is a coordinate. Such branches cross the imaginary axis for $k = \pm\infty$ at the imaginary roots of $Q_{i,i}(s) + sQ_{i,i-1}(s)$, if any, and for $k = -1$ at the imaginary roots of $Q_{i,i-1}(s)$, if any. In fact, both the even and the odd part of the left–hand side of (3.26) are equal to zero when $Q_{i,i-1}(s) = 0$ and $k = -1$. One locus branch passes through the point at infinity for

$$k = -\frac{a_{i,i}}{a_{i,i} + a_{i,i-1}} \tag{3.28}$$

since then the degree of (3.26) becomes $i - 1$. A typical root locus for (3.26) is shown in Fig. 3.3.

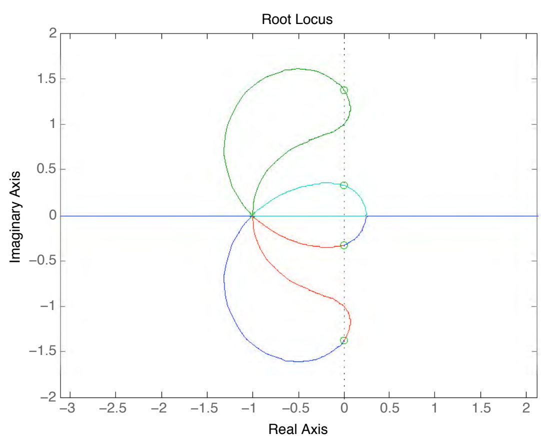

Figure 3.3: Root locus for (3.26) with $P_i(s) = P_4(s) = (s + 1)^4$.

On the basis of the previous considerations, the following result, whose proof can be found in [17], holds.

Theorem 3.4.1 *The polynomial $P_i(s)$ is Hurwitz and $-1 < r_{i,1} \leq 0$ if and only if $P_{i-1}(s)$ is Hurwitz and*

$$0 \geq r_{i,1} \geq r_{i,c} := -\frac{a_{i-1,i-1}}{a_{i-1,i-1} + a_{i-1,i-2}}, \tag{3.29}$$

$$-1 < f_{i,1} < f_{i,c} := \frac{r_{i,1} a_{i-1,i-1}}{(2 + r_{i,1}) a_{i-1,i-1} + 2 r_{i,1} a_{i-1,i-2}}. \tag{3.30}$$

□

A great saving in the computations required by the implementation of the considered stability–test procedures is obtained, in general, by resorting to their split forms [2], [3]. Since these recursive forms relate the even or odd parts of three consecutive polynomials in the corresponding sequences, they are called three–term or immittance–domain forms (see [7] concerning the z–domain Levinson algorithm). The interested reader is referred to [8] for a detailed analysis and classification of all such forms.

3.5 Concluding remarks

The classic stability–test algorithms have been expressed in their two–term recursive form. Then, a general recurrence relation capable of generating both the Routh–like and the Lepschy–like algorithms has been derived. The constraints that the parameters of this general recursion must satisfy to produce a stability–test algorithm have been pointed out.

Three main families of algorithms have been distinguished according to the shape of the root loci in which their two–term recursions can be embedded. Using this geometric approach, the stability criteria as well as the rules for counting the number of RHP and LHP roots of a given polynomial can be proved in a suggestive way. However, the two–term form of these algorithms is not the most efficient from the computational point of view.

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Often, short papers tend to be sharper than longer works because they focus on a single theme without lingering on unessential aspects, thus showing clearly the signicance of a contribution or an idea. The author of this book had the privilege of collaborating for over a quarter of a century with Antonio Lepschy (1931-2005), a recognized leader of the Italian control community.

Lepschy had a liking for the brief paper format, so that many results obtained by his research team were published in this way. The present compilation tells a few of these short stories, duly updated, trying to preserve their original flavour.

Umberto Viaro (http://umbertoviaro.blogspot.com/) has been professor of System and Control Theory at the University of Udine, Italy, since 1994. His 25-year-long collaboration with Antonio Lepschy resulted in more than 100 joint papers and two books. An essential role in this research activity was played by Wiesław Krajewski of the Systems Research Institute, Polish Academy of Sciences. The current research interests of Umberto Viaro concern optimal model reduction, robust control, switching and LPV control. He is the author or coauthor of 4 books and about 180 research papers.

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