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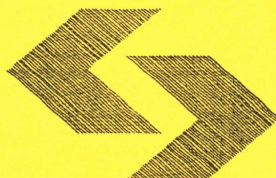
Research Report

**Order-Lipschitzian Properties
of Multifunctions
with Applications to Stability
of Minimal Points**

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Order-Lipschitzian properties of multifunctions with applications to stability of minimal points

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Abstract. We introduce order continuities of multifunctions. We apply the defined notions to stability of minimal points in Banach lattices.

1 Introduction

Lipschitz stability of efficient points requires some strong assumptions concerning order generating cone \mathcal{K} , see eg [1, 2, 3]. The appearance of these strong assumptions, like nonemptiness of the interior of \mathcal{K} , or boundedness of its basis, is motivated mainly by the fact that the classical definitions of continuity of set-valued mappings has purely topological character and are not related to the order structure of the space.

The idea of the present paper is to define order-Lipschitz continuity of set-valued mappings by exploiting as much as possible the order structure of the space and to derive sufficient conditions for the efficient points to be order-Lipschitz. In the literature there exists some papers devoted to similar attempts, eg [12, 8, 5].

Throughout the paper we assume that U and Y are an order complete Banach lattices. Recall that a space Y is order-complete if every nonempty subset of Y that is majorized in Y has a supremum. This is, in fact, equivalent to saying that every minorized subset of Y has an infimum. Recall that an ordered vector space Y is a vector lattice if $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist. A subset A of a vector lattice is called solid if $x \in A$, $y \in E$ and $|y| \leq |x|$ implies $y \in A$. A topological vector lattice Y is a vec-

tor lattice and a Hausdorff topological vector space (over \mathcal{R}) which possesses a base of solid 0-neighbourhoods. A Banach lattice Y is a normed vector lattice $(Y, \|\cdot\|)$ which is norm complete. For any lattice norm, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.

2 Basic definitions

Let $\mathcal{K}_+ \subset Y$ be the positive cone in a Banach lattice Y , $\mathcal{K}_+ = \{y \in Y \mid y \geq 0\}$. \mathcal{K} is clearly convex, closed and normal.

Definition 1 $\Gamma : U \rightrightarrows Y$ is said to be locally upper order Lipschitz, shortly l.u.o-Lipschitz, at u_0 if and only if there exist an open neighbourhood U_0 of u_0 and a $\ell \in \mathcal{K}_+$ such that for each $y \in \Gamma(u)$, $u \in U$, there exists $y_0 \in \Gamma(u_0)$ such that

$$|y - y_0| \leq \ell \|u - u_0\|. \quad (1)$$

By Banach lattice operations, upper local order-Lipschitzian property implies upper local norm Lipschitzian property.

By properties of the modulus

$$-\ell \|u - u_0\| \leq y - y_0 \leq +\ell \|u - u_0\|.$$

and $y \leq y_0 + \ell \|u - u_0\|$ and $y_0 - \ell \|u - u_0\| \leq y$. Consequently

$$\Gamma(u) \subset \Gamma(u_0) + \ell \|u - u_0\| - \mathcal{K} \quad \text{and} \quad \Gamma(u) \subset \Gamma(u_0) - \ell \|u - u_0\| + \mathcal{K}.$$

Example 1 Let $\mathcal{K} \subset \mathcal{R}^3$,

$$\mathcal{K} = \{(x, y, z) \mid z = 0, x, y \geq 0\}.$$

Let $\Gamma : \mathcal{R} \rightrightarrows \mathcal{R}^3$ be given as

$$\Gamma(0) = \{(x, y, z) \mid z = 0, 0 \leq x \leq 1, 0 \leq y \leq 1\}, \quad \Gamma(u) = \Gamma(0) \cup \{(1, 1, u)\}.$$

Γ is Lipschitz at 0 in the usual sense but not order Lipschitz.

Definition 2 $\Gamma : U \rightrightarrows Y$ is said to be locally lower order Lipschitz, shortly l.l.o-Lipschitz at u_0 if and only if there exist an open neighbourhood U_0 of u_0 and a $\ell \in \mathcal{K}_+$ such that for each $y_0 \in \Gamma(u_0)$, $u \in U$, there exists $y \in \Gamma(u)$, $u \in U$, such that

$$|y - y_0| \leq \ell \|u - u_0\|. \quad (2)$$

By Banach lattice operations, lower local order-Lipschitzian property implies lower local norm Lipschitzian property.

By properties of the modulus

$$-\ell\|u - u_0\| \leq y - y_0 \leq +\ell\|u - u_0\|.$$

and $y \leq y_0 + \ell\|u - u_0\|$ and $y_0 - \ell\|u - u_0\| \leq y$. Consequently

$$\Gamma(u_0) \subset \Gamma(u) + \ell\|u - u_0\| - \mathcal{K} \quad \text{and} \quad \Gamma(u_0) \subset \Gamma(u) - \ell\|u - u_0\| + \mathcal{K},$$

Finally if Γ is locally upper and lower order Lipschitz at u_0 it satisfies the following relations

$$\begin{aligned} \Gamma(u) \subset \Gamma(u_0) + \ell\|u - u_0\| - \mathcal{K} \quad \text{and} \quad \Gamma(u) \subset \Gamma(u_0) - \ell\|u - u_0\| + \mathcal{K} \\ \Gamma(u_0) \subset \Gamma(u) + \ell\|u - u_0\| - \mathcal{K} \quad \text{and} \quad \Gamma(u_0) \subset \Gamma(u) - \ell\|u - u_0\| + \mathcal{K} \end{aligned} \quad (3)$$

Proposition 1 *If Γ is order upper Lipschitz at u_0 , then any $y \in \Gamma(u)$ has a representation $y = y_0 + \ell\|u - u_0\| - k_y$, $y_0 \in \Gamma(u_0)$, $k_y \in \mathcal{K}$, and $\|k_y\| \leq 2\|\ell\|\|u - u_0\|$.*

Proof. By (3), we have

$$y = y_0 + \ell\|u - u_0\| - k_y, \quad y_0 \in \Gamma(u_0), \quad k_y \in \mathcal{K},$$

and by properties of Banach lattice operations

$$\|\ell\|\|u - u_0\| \geq \|y - y_0\| = \|k_y - \ell\|u - u_0\| \geq \|k_y\| - \|\ell\|\|u - u_0\|,$$

which gives

$$2\|\ell\|\|u - u_0\| \geq \|k_y\|.$$

□

Proposition 2 *If Γ is order upper Lipschitz at u_0 , then any $y \in \Gamma(u)$ has a representation $y = y_0 + \ell\|u - u_0\| - k_y$, $y_0 \in \Gamma(u_0)$, $k_y \in \mathcal{K}$, and $|k_y| \leq 2\ell\|u - u_0\|$.*

Proof. We have

$$k_y = y_0 - y + \ell\|u - u_0\|, \quad \text{and} \quad |k_y| \leq 2\ell\|u - u_0\|.$$

In consequence, $\|k_y\| \leq 2\|\ell\|\|u - u_0\|$.

□

3 Order-Lipschitz continuity of efficient points

Let $A \subset Y$ be a subset of Y . An element $y \in A$ is *efficient*, $y \in \text{Eff}(A, \mathcal{K}_+)$, if

$$(A - y) \cap (-\mathcal{K}_+) = \{0\}.$$

Let $\ell \in \mathcal{K}_+$. We denote

$$A_-(\ell) = A \setminus (\text{Eff}(A, \mathcal{K}_+) + \ell - \mathcal{K})$$

$$A_+(\ell) = A \setminus (\text{Eff}(A, \mathcal{K}_+) - \ell + \mathcal{K})$$

Definition 3 A set A has an ℓ -order containment property, ℓ -(OCP), if for each $\varepsilon > 0$ there exists $\delta > 0$ satisfying:

C1 for each $y \in A_-(\varepsilon\ell)$ there exists $\eta_y \in \text{Eff}(A, \mathcal{K}_+)$ satisfying

$$y - \eta_y - k \in \mathcal{K} \quad (4)$$

for all $k \in \mathcal{K}$, $k \leq \delta\ell$,

C2 for each $y \in A_+(\varepsilon\ell)$ there exists $\eta_y \in \text{Eff}(A, \mathcal{K}_+)$ satisfying

$$y - \eta_y - k \in \mathcal{K} \quad (5)$$

for all $k \in \mathcal{K}$, $k \leq \delta\ell$.

By (5), $y - \eta_y \geq k$, ie., $|y - \eta_y| \geq |k|$, and consequently $\|y - \eta_y\| \geq \delta\|\ell\|$.

If (C1) holds for A , then

$$A \subset \text{cl}[\text{Eff}(A, \mathcal{K}_+) - \mathcal{K}_+] \cup [\text{Eff}(A, \mathcal{K}_+) + \mathcal{K}_+]. \quad (6)$$

Indeed, suppose that (C1) holds. If $y \in A \setminus \text{cl}[\text{Eff}(A, \mathcal{K}_+) - \mathcal{K}_+]$, then $y \notin \text{Eff}(A, \mathcal{K}_+) + \varepsilon\ell - \mathcal{K}_+$ because $\text{cl}[\text{Eff}(A, \mathcal{K}_+) - \mathcal{K}_+]$ is closed, and, by (5), $y \in \text{Eff}(A, \mathcal{K}_+) + \mathcal{K}_+$. This proves (6). If $\text{cl}\text{Eff}(A, \mathcal{K}_+)$ is compact, then (6) takes the form

$$A \subset [\text{cl}\text{Eff}(A, \mathcal{K}_+) - \mathcal{K}_+] \cup [\text{Eff}(A, \mathcal{K}_+) + \mathcal{K}_+] \quad (7)$$

because $\text{clEff}(A, \mathcal{K}_+) - \mathcal{K}_+$ is closed. If $\text{Eff}(A, \mathcal{K}_+)$ is compact, then (6) takes the form

$$A \subset \text{Eff}(A, \mathcal{K}_+) + \mathcal{K}_+. \quad (8)$$

Let $\mathcal{M} : U \rightrightarrows Y$ be a set-valued mapping defines

$$\mathcal{M}(u) = \text{Eff}(\Gamma(u), \mathcal{K}_+).$$

Theorem 1 *Let Y , U be Banach lattices. Let $\mathcal{K}_+ \subset Y$ be a positive cone in Y . Assume that*

- (i) Γ is order upper Lipschitz at u_0 , and order lower Lipschitz at u_0 ,
- (ii) ℓ -order containment property holds for $\Gamma(u_0)$ with rate $\delta(\varepsilon) \geq 2c\varepsilon$, $c > 0$.

The minimal point multifunction \mathcal{M} is order upper Lipschitz at u_0 .

Proof. By (i),

$$\begin{aligned} \Gamma(u) &\subset [\text{Eff}(\Gamma(u_0), \mathcal{K}_+) + \frac{\ell}{c}\|u - u_0\| + \ell\|u - u_0\| - \mathcal{K}_+] \cup \\ &\Gamma(u_0) \setminus (\text{Eff}(\Gamma(u_0), \mathcal{K}_+) + \frac{\ell}{c}\|u - u_0\| - \mathcal{K}_+) + \ell\|u - u_0\| - \mathcal{K}_+. \end{aligned}$$

We show that if $y \in \Gamma(u) \cap [\Gamma(u_0) \setminus (\text{Eff}(\Gamma(u_0), \mathcal{K}_+) + \frac{\ell}{c}\|u - u_0\| - \mathcal{K}_+) + \ell\|u - u_0\| - \mathcal{K}_+]$, then $y \notin \text{Eff}(\Gamma(u), \mathcal{K}_+)$.

Indeed, let $y \in \Gamma(u)$ and

$$y \in [\Gamma(u_0) \setminus (\text{Eff}(\Gamma(u_0), \mathcal{K}_+) + \frac{7}{c}\|u - u_0\| - \mathcal{K}_+)] + \ell\|u - u_0\| - \mathcal{K}_+.$$

Then $y = \gamma + \ell\|u - u_0\| - k_y$, $\gamma \in \Gamma(u_0) \setminus (\text{Eff}(\Gamma(u_0), \mathcal{K}_+) + \frac{\ell}{c}\|u - u_0\| - \mathcal{K}_+)$, $k_y \in \mathcal{K}_+$. By Proposition 2, $k_y \leq 2\ell\|u - u_0\|$.

By (C1), there exists $\eta_\gamma \in \text{Eff}(\Gamma(u_0), \mathcal{K}_+)$ such that

$$\gamma - \eta_\gamma - k \in \mathcal{K}_+ \quad \text{for all } k \in \mathcal{K}_+, k \leq \delta(\frac{7}{c}\|\ell\|\|u - u_0\|)\ell \quad (9)$$

By the lower order Lipschitz continuity of Γ at u_0 we have

$$\eta_\gamma = z + \ell\|u - u_0\| - k_z, \quad z \in \Gamma(u), \quad k_z \in \mathcal{K}_+, \quad k_z \leq 2\ell\|u - u_0\|.$$

In consequence,

$$y - z = [\gamma - \eta_\gamma] + \ell\|u - u_0\| - k_z + \ell\|u - u_0\| - k_y$$

and by (10), since $\|k_z + k_y\| \leq 4\|\ell\|\|u - u_0\| \leq \delta(7\|\ell\|\|u - u_0\|)$

$$y - z \in \mathcal{K} \setminus \{0\},$$

because $\|\gamma - \eta_\gamma\| \geq 7\|\ell\|\|u - u_0\|$, and $\|\ell\|\|u - u_0\| - k_z + \ell\|u - u_0\| - k_y \leq 6\|\ell\|\|u - u_0\|$.

By (i),

$$\begin{aligned} \Gamma(u) &\subset [\text{Eff}(\Gamma(u_0), \mathcal{K}_+) - \frac{5}{c}\ell\|u - u_0\| - \ell\|u - u_0\| + \mathcal{K}] \cup \\ &\Gamma(u_0) \setminus (\text{Eff}(\Gamma(u_0), \mathcal{K}_+) - \frac{5}{c}\ell\|u - u_0\| + \mathcal{K}) - \ell\|u - u_0\| + \mathcal{K}. \end{aligned}$$

We show that if $y \in \Gamma(u) \cap [\Gamma(u_0) \setminus (\text{Eff}(\Gamma(u_0), \mathcal{K}_+) - \frac{5}{c}\ell\|u - u_0\| + \mathcal{K}_+) - \ell\|u - u_0\| + \mathcal{K}_+]$, then $y \notin \text{Eff}(\Gamma(u), \mathcal{K}_+)$.

Indeed, let $y \in \Gamma(u)$ and

$$y \in [\Gamma(u_0) \setminus (\text{Eff}(\Gamma(u_0), \mathcal{K}_+) - \frac{5}{c}\ell\|u - u_0\| + \mathcal{K}_+)] - \ell\|u - u_0\| + \mathcal{K}_+.$$

Then $y = \gamma - \ell\|u - u_0\| + k_y$, $\gamma \in \Gamma(u_0) \setminus (\text{Eff}(\Gamma(u_0), \mathcal{K}_+) - \frac{5}{c}\ell\|u - u_0\| + \mathcal{K}_+)$, $k_y \in \mathcal{K}_+$. By Proposition 2, $k_y \leq 2\ell\|u - u_0\|$.

By (C2), there exists $\eta_\gamma \in \text{Eff}(\Gamma(u_0), \mathcal{K}_+)$ such that

$$\gamma - \eta_\gamma - k \in \mathcal{K}_+ \quad \text{for all } k \in \mathcal{K}_+, \quad k \leq \delta(\frac{5}{c}\|\ell\|\|u - u_0\|)\ell \quad (10)$$

By the lower order Lipschitz continuity of Γ at u_0 we have

$$\eta_\gamma = z - \ell\|u - u_0\| + k_z, \quad z \in \Gamma(u), \quad k_z \in \mathcal{K}_+, \quad k_z \leq 2\ell\|u - u_0\|, \quad z = \eta_\gamma + \ell\|u - u_0\| - k_z.$$

In consequence,

$$y - z = [\gamma - \eta_\gamma] - \ell\|u - u_0\| + k_z - \ell\|u - u_0\| + k_y$$

and by (10), since $\|k_z + k_y\| \leq 4\|\ell\|\|u - u_0\| \leq \delta(7\|\ell\|\|u - u_0\|)$

$$y - z \in \mathcal{K}_+ \setminus \{0\},$$

because $\|\gamma - \eta_\gamma\| \geq 7\|\ell\|\|u - u_0\|$, and $\|\ell\|\|u - u_0\| - k_z + \ell\|u - u_0\| - k_y \leq 6\|\ell\|\|u - u_0\|$.

□

4 Conclusions

The definition of order-Lipschitz continuity of set-valued mappings introduced above is, in general, stronger than the usual Lipschitz continuity. For instance, in finite-dimensional case, roughly speaking, it allows Γ to vary only in varieties parallel to $\text{aff}\mathcal{K}_+$. On the other hand, we derive sufficient conditions for efficient points to be order-Lipschitz without any additional requirements on \mathcal{K}_+ .

Let us note that both upper and lower order-Lipschitz continuities introduced above are characterised by pairs of inclusions which do not involve the lattice structure of the space. Hence, those pairs of inclusions give rise to definitions of order-Lipschitzian properties in more general spaces, and, finally, each inclusion separately can also be viewed as a kind of Lipschitzian property and the respective sufficient conditions for such property to hold for efficient points are contained in those proved by Theorem 1.

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