These results agree with those obtained in the previous article, if $n = n'$.

Now let us see with what relation between $n$ and $n'$ it is possible for $R_1$ to equal $R_4$, whatever $\gamma$ may be, *i.e.* even if it be small. We must have

$$10 - \frac{48}{n'}\gamma = 3 - \frac{48}{n}\gamma,$$

or

$$\frac{7}{48\gamma} = \frac{1}{n'} - \frac{1}{n}.$$

Thus $n'$ must be less than $n$, or the middle column must be of greater cross-section than the terminal columns. Further, since $\gamma$ is usually very small, $n'$ must be *very much* less than $n$, or approximately

$$n' = \tfrac{48}{7}\gamma\left\{1 - \frac{48}{7n}\gamma\right\}.$$

Hence, by putting a very massive mid-column of otherwise arbitrary cross-section, and two very slender terminal columns with cross-sections about $\frac{48}{7}\gamma$ times that of the girder itself, we should have the three support-reactions nearly equal. Dorna's theory makes them equal when the three columns are of equal cross-section, which is obviously impossible for any reasonable value of $\gamma$.

University College, London,
*Dec.* 21*st*, 1889.

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# ON LATIN SQUARES.

By Prof. *Cayley*.

If in each line of a square of $n^2$ compartments the same $n$ letters $a, b, c, \ldots$ are arranged so that no letter occurs twice in the same column, we have what was termed by Euler "a Latin square." Supposing that in one of the lines the letters are arranged in the natural order *abcde*..., then in the remaining lines there must be arrangements beginning with $b, c, d, e$, &c., respectively, and we may consider the case in which the bottom line has the arrangement *abcde*..., and in the other lines, reckoning from the bottom one in order, the arrangements begin with $b, c, d, e$, &c., respectively: if the number of such squares be $= N$, then, obviously, the whole number of squares which can be formed with the same $n$ arrangements is $= N[n]^n$.

Starting with the bottom line as above, then it is a well-known problem to determine the number of arrangements for the second line, viz., this number is

$$= 1.2.3\ldots n \left\{1 - \frac{1}{1} + \frac{1}{1.2} \ldots \pm \frac{1}{1.2\ldots n}\right\};$$

and if we assume, as above, that the second line begins with $b$, then the whole number of arrangements is this number divided by $(n-1)$, the quotient being of course integral. For instance, $n=5$, the number is $= 120 - 120 + 60 - 20 + 5 - 1, = 44$, which is divisible by 4, and the number of arrangements for the second line is thus $= 11$.

But the number of arrangements for the third line will be different according to the arrangement selected for the second line, and it is not easy to see how in general the whole number of arrangements for the third line is to be calculated, and the difficulty of course increases for the next following lines; it may be remarked that, when all except the top-line are filled up, the top-line is completely determined.

Imagine the square completed, we may write down the substitutions by which we pass from the bottom line to itself (this is of course the substitution 1) and to each of the other lines respectively; we have thus a set of $n$ substitutions, which may form *a group*; and when this is so, we may conversely from the group construct the latin square. But it is not every Latin square which is thus connected with a group of $n$ substitutions.

In the cases $n = 2, 3, 4$ there is no difficulty, the squares are

| | | | | | |
|---|---|---|---|---|---|
| *b a*, | *c a b*, | *d c a b*, | *d c b a*, | *d c b a*, | *d a b c*, |
| *a b* | *b c a* | *c d b a* | *c d a b* | *c a d b* | *c d a b* |
| | *a b c* | *b a d c* | *b a d c* | *b d a c* | *b c d a* |
| | | *a b c d* | *a b c d* | *a b c d* | *a b c d* |

viz., $n=2$ the number is 1, $n=3$ it is 1, $n=4$ it is 4; in this last case the arrangement *badc* for the second line gives two squares, but each of the other arrangements only one square.

In each of the squares of 4 we have a group, viz., for the four squares respectively, these are

| | | | | |
|---|---|---|---|---|
| 1°, | 1, | $(ab)(cd)$, | $(acbd)$, | $(adbc)$, |
| 2 | 1 | $(ab)(cd)$ | $(ac)(bd)$ | $(ad)(bc)$ |
| 3° | 1 | $(abdc)$ | $(acdb)$ | $(ad)(bc)$ |
| 4° | 1 | $(abcd)$ | $(ac)(bd)$ | $(adcb)$ |

1°, 3°, 4° are the cyclical groups of $(acbd)$, $(abdc)$, and $(abcd)$, respectively; 2° is a different kind of group.

In the case $n=5$, the whole number of squares is 56, viz., there are five arrangements of the second line each giving four squares, and six arrangements each giving six squares, $5.4+6.6=56$. The five arrangements are

| | | | | |
|---|---|---|---|---|
| *b a e c d*, | *b a d e c*, | *b c a e d*, | *b d e a c*, | *b e d c a*, |
| *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* |

viz., in these cases, the substitutions for passing to the second line are $(ab)(ced)$, $(ab)(cde)$, $(abc)(de)$, $(abd)(ce)$, $(abe)(cd)$, respectively.

The six arrangements are

| | | | | | |
|---|---|---|---|---|---|
| *b d a e c*, | *b e a c d*, | *b c e a d*, | *b e d a c*, | *b c d e a*, | *b d e c a*, |
| *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* |

viz., in these cases, the substitutions for passing to the second line are $(abdec)$, $(abedc)$, $(abced)$, $(abecd)$, $(abcde)$, $(abdce)$, respectively.

A set of four squares is

| | | | |
|---|---|---|---|
| *e c d b a*, | *e d a b c*, | *e c d a b*, | *e d b a c*, |
| *d e b a c* | *d c b e a* | *d e a b c* | *d c a e b* |
| *c d a e b* | *c e d a b* | *c d b e a* | *c e d b a* |
| *b a e c d* | *b a e c d* | *b a e c d* | *b a e c d* |
| *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* |

and a set of six squares is

| | | | | | |
|---|---|---|---|---|---|
| *e c d a b*, | *e a d c b*, | *e c d b a*, | *e a b c d*, | *e a b c d*, | *e c b a d*. |
| *d e b c a* | *d c e b a* | *d a e c b* | *d c e b a* | *d c e a b* | *d a e c b* |
| *c a e b d* | *c e b a d* | *c e b a d* | *c e d a b* | *c e d b a* | *c e d b a* |
| *b d a e c* | *b d a e c* | *b d a e c* | *b d a e c* | *b d e a c* | *b d a e c* |
| *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* | *a b c d e* |

In a square belonging to a set of four, the substitutions for obtaining from any one line all the other lines are of a form such as 1, $(ab)(ced)$, $(ac)(bde)$, $(ad)(bec)$, $(ae)(bcd)$, which are not a group. In the case of a set of six squares, there is one square of the set (in the foregoing instance the first square) where the substitutions are of a form such as 1, $(abdec)$, $(acedb)$, $(adceb)$, $(aebcd)$, and which thus form a cyclical group of five substitutions.