ON THE STABILITY OF A BENT AND TWISTED WIRE.

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If a wire of isotropic section and naturally straight be twisted, and the two ends joined so as to form a continuous curve, the circle will be a stable form of equilibrium for less than a certain amount of twist.*

I propose in this note to determine the limit of stability. I begin by finding the general intrinsic equations of vibration of a bent wire.

Let AB be an element of the wire bounded by normal sections A, B, and let the distances of these sections from a fixed point of the wire be $s - \delta s$ and s respectively.

Let S, T, U be the components of the resultant force on the section B, S being measured along the tangent in the direction of s increasing, T along the principal normal inwards, and U along the binormal, so that the three directions form a right-handed system.

Let F, G, H be the components of the couple on the section B in the same three directions.

Then, if P, Q, R are the impressed forces on the element AB per unit length, the equations of equilibrium are

$$\left.\begin{aligned} \frac{dS}{ds} - \kappa T + \quad P \quad &= 0 \\ \frac{dT}{ds} - \tau U + \kappa S + Q &= 0 \\ \frac{dU}{ds} + \tau T + \quad R \quad &= 0 \\ \frac{dF}{ds} - \kappa G \qquad\qquad &= 0 \\ \frac{dG}{ds} - \tau H + \kappa F - U &= 0 \\ \frac{dH}{ds} + \tau G + \quad T \quad &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where κ is the curvature and τ the torsion at s.

* Thomson and Tait, *Nat. Phil.*, § 123.

Now let γ be the rate of twist of the wire at s, then the theory of wires gives

$$\left.\begin{aligned} F &= M\gamma \\ G &= 0 \\ H &= L\kappa \end{aligned}\right\},$$

assuming the flexibility the same in all directions.

Substituting in equations (1), we get

$$\frac{dF}{ds} = 0,$$

$$U = -L\,\kappa\tau + M\,\kappa\gamma,$$

$$T = -L\frac{d\kappa}{ds},$$

so that the twist γ is constant, and

$$S = -\frac{Q}{\kappa} - \tau(L\tau - M\gamma) + L\frac{1}{\kappa}\frac{d^2\kappa}{ds^2}.$$

Substituting in the two remaining equations of (1), we get the two dynamical equations

$$\left.\begin{aligned} L\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2\kappa}{ds^2} + \tfrac{1}{2}\kappa^2 - \tau^2\right) + M\gamma\frac{d\tau}{ds} &= -P + \frac{d}{ds}\frac{Q}{\kappa} \\ L\left(\frac{d}{ds}\kappa\tau + \tau\frac{d\kappa}{ds}\right) - M\gamma\frac{d\kappa}{ds} &= R \end{aligned}\right\} \ldots(2).$$

Now let the wire vibrate about its equilibrium form, and let u, v, w be the displacements of the point s along the tangent, principal normal, and binormal respectively at time t.

Supposing no impressed forces we have

$$-P = m\frac{d^2u}{dt^2},$$

$$-Q = m\frac{d^2v}{dt^2},$$

$$-R = m\frac{d^2w}{dt^2},$$

and the condition of inextensibility is

$$\frac{du}{ds} = \kappa v,$$

so that

$$-Q = \frac{m}{\kappa}\frac{d^3u}{ds\,dt^2}.$$

Further, if κ_0, τ_0 denote the equilibrium values of the curvature and torsion respectively, we have*

$$\left.\begin{aligned} \kappa - \kappa_0 &= \frac{d\alpha}{ds} - \tau\beta \\ \tau - \tau_0 &= \frac{d}{ds}\frac{1}{\kappa}\frac{d\beta}{ds} + \kappa\beta + \frac{d}{ds}\frac{\tau}{\kappa}\alpha \end{aligned}\right\} \text{...........(3)},$$

where
$$\alpha = \frac{d}{ds}\frac{1}{\kappa}\frac{du}{ds} + \kappa u - \tau w,$$

$$\beta = \frac{dw}{ds} + \frac{\tau}{\kappa}\frac{du}{ds}.$$

Substituting these values in equations (2), we have the general intrinsic equations of vibration.

When the equilibrium form is a plane curve, these equations reduce to

$$\left.\begin{aligned} L\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2\kappa}{ds^2} + \tfrac{1}{2}\kappa^2\right) + M\gamma\frac{d\tau}{ds} &= m\frac{d^2u}{dt^2} - m\frac{d}{ds}\frac{1}{\kappa^2}\frac{d^3u}{ds\,dt^2} \\ L\left(\frac{d}{ds}\kappa\tau + \tau\frac{d\kappa}{ds}\right) - M\gamma\frac{d\kappa}{ds} &= -m\frac{d^2w}{dt^2} \end{aligned}\right\},$$

where
$$\kappa = \kappa_0 + \frac{d^2}{ds^2}\frac{1}{\kappa}\frac{du}{ds} + \frac{d}{ds}\kappa u,$$

$$\tau = \frac{d}{ds}\frac{1}{\kappa}\frac{d^2w}{ds^2} + \kappa\frac{dw}{ds}.$$

Proceeding to the particular case of a circular ring, the equations are

$$L\frac{1}{\kappa^2}\left(\frac{d^3}{ds^3} + \kappa^2\frac{d}{ds}\right)^2 u + M\gamma\frac{1}{\kappa}\left(\frac{d^4}{ds^4} + \kappa^2\frac{d^2}{ds^2}\right)w = m\left(\frac{d^2u}{dt^2} - \frac{1}{\kappa^2}\frac{d^4u}{ds^2dt^2}\right),$$

$$-L\left(\frac{d^4}{ds^4} + \kappa^2\frac{d^2}{ds^2}\right)w + M\gamma\frac{1}{\kappa}\left(\frac{d^4}{ds^4} + \kappa^2\frac{d^2}{ds^2}\right)u = m\frac{d^2w}{dt^2}.$$

The appropriate solution, when the wire forms a complete circle, is

$$u = Ae^{i(pt - rs\kappa)},$$
$$w = Be^{i(pt - rs\kappa)},$$

r being an integer.

* "The small deformation of curves and surfaces, &c.," *ante* p. 68.

Making the substitutions and eliminating A, B, we find

$$\begin{vmatrix} mp^2(1+r^2)-L\kappa^4r^2(1-r^2)^2, & -M\gamma\kappa^3r^2(1-r^2) \\ -M\gamma\kappa^3r^2(1-r^2), & mp^2+L\kappa^4r^2(1-r^2) \end{vmatrix}=0,$$

or

$$m^2p^4(1+r^2)+2mp^2L\kappa^4r^4(1-r^2)-L^2\kappa^8r^4(1-r^2)^3$$
$$-M^2\gamma^2\kappa^6r^4(1-r^2)^2=0.$$

For stability, the values of p^2 must be positive, and this leads to the condition

$$L^2\kappa^2(r^2-1)>M^2\gamma^2,$$

Now $r=1$ corresponds merely to displacement of the ring as a rigid body.

The necessary condition for stability is therefore

$$\frac{\gamma}{\kappa}<\frac{L}{M}\sqrt{(3)},$$

so that the total twist must be less than

$$2\sqrt{(3)}\,\pi L/M.$$

If the cross-section is circular,

$$\frac{L}{M}=\frac{E}{2\mu},$$

where E is Young's modulus and μ is the rigidity modulus.

For metals $E=\frac{5}{2}\mu$ about, and in this case the total twist must be less than $2\pi\times2.16$.

ON THE EQUATION $x^{17}-1=0$.

By Prof. *Cayley*.

WRITING $\rho=\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{7}$, I carry the solution up to the determination of the periods each of two roots, $\rho+\rho^{16}$, $=2\cos\frac{2\pi}{17}$, &c. The expressions contain the radicals

$$a=\sqrt{(17)},\ b=\sqrt{\{2(17-a)\}},\ c=\sqrt{\{4(17+3a)-2(3+a)b\}},$$

where a, b, c are taken to be positive ($a=4{\cdot}12$, $b=5{\cdot}07$, $c=6{\cdot}72$). Taking for a moment r to be any imaginary seventeenth root, $r=\rho^\theta$, then the algebraical expression for the period P_1 of eight roots is $P_1=\frac{1}{2}(-1\pm a)$, but I assume the value to be $P_1=\frac{1}{2}(-1+a)$, and thus determine θ to denote some one of