Making the substitutions and eliminating A, B, we find

$$\begin{vmatrix} mp^2(1+r^2)-L\kappa^4r^2(1-r^2)^2, & -M\gamma\kappa^3r^2(1-r^2) \\ -M\gamma\kappa^3r^2(1-r^2), & mp^2+L\kappa^4r^2(1-r^2) \end{vmatrix}=0,$$

or

$$m^2p^4(1+r^2)+2mp^2L\kappa^4r^4(1-r^2)-L^2\kappa^8r^4(1-r^2)^3 \\ -M^2\gamma^2\kappa^6r^4(1-r^2)^2=0.$$

For stability, the values of p^2 must be positive, and this leads to the condition

$$L^2\kappa^2(r^2-1)>M^2\gamma^2,$$

Now $r=1$ corresponds merely to displacement of the ring as a rigid body.

The necessary condition for stability is therefore

$$\frac{\gamma}{\kappa}<\frac{L}{M}\surd(3),$$

so that the total twist must be less than

$$2\surd(3)\,\pi\,L/M.$$

If the cross-section is circular,

$$\frac{L}{M}=\frac{E}{2\mu},$$

where E is Young's modulus and μ is the rigidity modulus.

For metals $E=\frac{5}{2}\mu$ about, and in this case the total twist must be less than $2\pi\times2.16$.

ON THE EQUATION $x^{17}-1=0$.

By Prof. *Cayley*.

Writing $\rho=\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{7}$, I carry the solution up to the determination of the periods each of two roots, $\rho+\rho^{16}$, $=2\cos\frac{2\pi}{17}$, &c. The expressions contain the radicals

$$a=\surd(17),\ b=\surd\{2(17-a)\},\ c=\surd\{4(17+3a)-2(3+a)b\},$$

where a, b, c are taken to be positive (a=4·12, b=5·07, c=6·72). Taking for a moment r to be any imaginary seventeenth root, $r=\rho^6$, then the algebraical expression for the period P_1 of eight roots is $P_1=\frac{1}{2}(-1\pm a)$, but I assume the value to be $P_1=\frac{1}{2}(-1+a)$, and thus determine θ to denote some one of the values 1, 2, 4, 8, 9, 13, 15, 16; similarly, I assume the value of Q_1 to be $=\frac{1}{4}(-1+a+b)$; and thus further determine θ to denote some one of the values 1, 4, 13, 16: and, again, I assume the value of R_1 to be $=\frac{1}{8}(-1+a+b+c)$, and thus further determine θ to denote one of the values 1 and 16. As regards the values of the periods R, it is obviously indifferent which value is taken, and I assume therefore $\theta=1$. This comes to saying that the signs of the radicals are determined in suchwise that r shall denote the root $\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}$, and it is to be understood that r has this value.

I write now

$$
\begin{aligned}
P_1 &= r + r^9 + r^{13} + r^{15} + r^{16} + r^8 + r^4 + r^2,\\
P_2 &= r^3 + r^{10} + r^5 + r^{11} + r^{14} + r^7 + r^{12} + r^6,\\
Q_1 &= r + r^{13} + r^{16} + r^4,\\
Q_2 &= r^9 + r^{15} + r^8 + r^2,\\
Q_3 &= r^3 + r^5 + r^{14} + r^{12},\\
Q_4 &= r^{10} + r^{11} + r^7 + r^6,\\
R_1 &= r + r^{16},\\
R_2 &= r^{13} + r^4,\\
R_3 &= r^9 + r^8,\\
R_4 &= r^{15} + r^2,\\
R_5 &= r^3 + r^{14},\\
R_6 &= r^5 + r^{12},\\
R_7 &= r^{10} + r^7,\\
R_8 &= r^{11} + r^6,
\end{aligned}
$$

and moreover

$$
\begin{aligned}
a &= P_1 - P_2,\\
b &= 2(Q_1 - Q_2),\\
+b_1 &= 2(Q_3 - Q_4),\\
c &= 4(R_1 - R_2),\\
-c_1 &= 4(R_3 - R_4),\\
+c_2 &= 4(R_5 - R_6),\\
-c_3 &= 4(R_7 - R_8).
\end{aligned}
$$

It will appear by what follows that a is determined by the quadric equation $a^2=17$, but I have assumed that a denotes the positive root $a=\surd(17)$; similarly b is determined by the quadric equation $b^2=2(17-a)$, but it is assumed that b denotes the positive root, $b=\surd\{2(17-a)\}$; and c is determined by the quadric equation $c^2=4(17+3a)-2(3+a)b$, but it is assumed that c denotes the positive root,

$$=\surd\{4(17+3a)-2(3+a)b\}.$$

If in the equations I had written b_1, c_1, c_2, c_3 (instead of $+b_1$, $-c_1$, $+c_2$, $-c_3$), then b_1 comes out rationally in terms of a, b; and c_1, c_2, c_3 come out rationally in terms of a, b, c; the signs were attached to them *à posteriori*, in suchwise that the values of b_1, c_1, c_2, c_3 might be each of them positive; for their independent determination, we have in fact for b_1^2 an expression such as that for b^2; and for c_1^2, c_2^2, c_3^2 expressions such as that for c^2; and taking as above for each of them the positive value of the square root, we have

$$\begin{aligned}
a &= \surd(17), && (=\ 4{\cdot}12),\\
b &= \surd\{2(17-a)\}, && (=\ 5{\cdot}07),\\
b_1 &= \surd\{2(17+a)\}, && (=\ 6{\cdot}49),\\
c &= \surd\{4(17+3a)-2(\ \ 3+a)b\}, && (=\ 6{\cdot}72),\\
c_1 &= \surd\{4(17+3a)+2(\ \ 3+a)b\}, && (=13{\cdot}77),\\
c_2 &= \surd\{4(17-3a)+2(-3+a)b_1\}, && (=\ 5{\cdot}75),\\
c_3 &= \surd\{4(17-3a)-2(-3+a)b_1\}, && (=\ 2{\cdot}02).
\end{aligned}$$

The relations between the periods P are

$$P_1+P_2=-1,$$

	P_1	P_2
P_1	$-P_1+4$	-4
P_2		$-P_2+4$

	a
a	17

that is $P_1^2=-P_1+4$, &c.; $a^2=17$.

Here for the second square we have

$$a^2=(P_1-P_2)^2=P_1^2+P_2^2-2P_1P_2=(-P_1+4)+(-P_2+4)+8,$$
$$=-P_1-P_2+16,=17;$$

and similarly in the other cases which follow.

For the periods Q, we have

$$Q_1+Q_2=P_1,$$
$$Q_3+Q_4=P_2,$$

	Q_1	Q_2	Q_3	Q_4
Q_1	Q_2+2Q_3+4	-1	$2Q_1+Q_2+Q_4$	$Q_2+Q_3+2Q_4$
Q_2		Q_1+2Q_4+4	Q_1+2Q_3+4	$Q_1+2Q_2+Q_3$
Q_3			$2Q_2+Q_4+4$	-1
Q_4				$2Q_1+Q_3+4$

	b	b_1
b	$34-2a$	$8a$
b_1		$34+2a$

so that $bb_1=8a$, or b_1 is given rationally in terms of a, b.

And for the periods R, we have

$$R_1+R_2=Q_1,$$
$$R_3+R_4=Q_2,$$
$$R_5+R_6=Q_3,$$
$$R_7+R_8=Q_4,$$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
R_1	R_4+2	R_5+R_6	R_3+R_7	R_1+R_5	R_2+R_4	R_2+R_8	R_3+R_8	R_6+R_7
R_2		R_3+2	R_2+R_6	R_4+R_8	R_1+R_7	R_1+R_3	R_5+R_8	R_4+R_7
R_3			R_1+2	R_7+R_8	R_6+R_8	R_2+R_5	R_1+R_4	R_4+R_5
R_4				R_2+2	R_1+R_6	R_5+R_7	R_3+R_6	R_2+R_3
R_5					R_8+2	R_3+R_4	R_2+R_7	R_3+R_5
R_6						R_7+2	R_4+R_6	R_1+R_8
R_7							R_5+2	R_1+R_2
R_8								R_3+2

where observe that the overlined terms R_5+R_6, R_7+R_8, R_3+R_4, and R_1+R_2, have the values Q_4, Q_3, Q_2, Q_1 respectively, and

	c	c_1	c_2	c_3
c	$4(17+3a)-2(3+a)b$	$8(b+b_1)$	$4(2a-b+b_1)$	$4(-2a+b+b_1)$
c_1		$4(17+3a)+2(3+a)b$	$4(2a+b+b_1)$	$4(2a+b-b_1)$
c_2			$4(17-3a)+2(-3+a)b_1$	$8(-b+b_1)$
c_3				$4(17-3a)-2(-3+a)b_1$

in which last table b_1 may be considered as denoting its value $=\frac{8a}{b}$, so that c_1, c_2, c_3 are each given rationally in terms of a, b, c.

And from the foregoing results, we have

$$\begin{aligned}
P_1 &= \tfrac{1}{2}(-1+a), &&= 1{\cdot}56,\\
P_2 &= \tfrac{1}{2}(-1-a), &&= -2{\cdot}56,\\
Q_1 &= \tfrac{1}{4}(-1+a+b), &&= 2{\cdot}05,\\
Q_2 &= \tfrac{1}{4}(-1+a-b), &&= -0{\cdot}49,\\
Q_3 &= \tfrac{1}{4}(-1-a+b_1), &&= 0{\cdot}34,\\
Q_4 &= \tfrac{1}{4}(-1-a-b_1), &&= -2{\cdot}90,\\
R_1 &= \tfrac{1}{8}(-1+a+b+c), &&= 1{\cdot}87,\\
R_2 &= \tfrac{1}{8}(-1+a+b-c), &&= 0{\cdot}18,\\
R_3 &= \tfrac{1}{8}(-1+a-b-c_1), &&= -1{\cdot}96,\\
R_4 &= \tfrac{1}{8}(-1+a-b+c_1), &&= 1{\cdot}47,\\
R_5 &= \tfrac{1}{8}(-1-a+b_1+c_2), &&= 0{\cdot}8\ ,\\
R_6 &= \tfrac{1}{8}(-1-a+b_1-c_2), &&= -0.55,\\
R_7 &= \tfrac{1}{8}(-1-a-b_1-c_3), &&= -1{\cdot}70,\\
R_8 &= \tfrac{1}{8}(-1-a-b_1+c_3), &&= -1{\cdot}20.
\end{aligned}$$

The approximate numerical values have been given throughout only for the purpose of showing that the signs of the square roots have been rightly determined.

QUATERNION PROOFS OF THEOREMS RELATING TO ASYMPTOTIC LINES.

By *T. Motoda.*

I HAVE proved in a simple way some theorems in Geometry, concerning asymptotic lines, by the use of the quaternion analysis as follows:

Let ρ be the vector of an asymptotic curve of the surface $S\nu d\sigma=0$; then $S\nu d\rho=0$, or ν is perpendicular to $d\rho$. Along the curve two consecutive normals of the surface are parallel,