MESSENGER OF MATHEMATICS.

ON ARITHMETICAL SERIES.

By Professor *Sylvester*.

THE first part of this article relates to the prime numbers (or so to say latent primes) contained as factors of the terms of given arithmetical series; the second part will deal with the actual or, say, visible primes included among such terms. Both investigations repose alike upon certain elementary theorems concerning the "index-sums" (relative to any given prime) of arithmetical series, whether simple and continuous as in the case of series ordinarily so called or compound and interstitial as such before named series become when subjected to certain periodic and uniform interruptions.

PART I.

§ 1. *Preliminary Notions.*

Consider any given sequence

$$m+1,\ m+2,\ m+3,\ \ldots,\ m+n,$$

in relation to any given prime number q.

Let r be the sum of the indices of the highest powers of q which are contained in the several terms of the natural sequence

$$1,\ 2,\ 3,\ \ldots,\ n,$$

s the sum of the indices of the highest powers of q contained in the given sequence.

Then it is almost immediately obvious that $s =$ or $> r$, *i.e.* $s > r - 1$.

For the index-sum of the natural sequence will be represented by

$$r = E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \cdots,$$

and the index-sum of the given sequence by

$$s = E\left(\frac{m+n}{q}\right) + E\left(\frac{m+n}{q^2}\right) + E\left(\frac{m+n}{q^3}\right) + \cdots$$
$$- E\left(\frac{m}{q}\right) - E\left(\frac{m}{q^2}\right) - E\left(\frac{m}{q^3}\right) - \cdots$$
$$= \text{or} > E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \cdots,$$

i.e. $s =$ or $> r$.

But there is another and more important theorem, less immediately obvious, and more germane to the subject-matter of the following section, which I proceed to explain.

Suppose $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ to be the several exponents of the highest powers of q which are contained in

$$x+1,\ x+2,\ x+3,\ \ldots,\ x+n,$$

and let σ be that one of these n exponents which is not less than any other of them.

Call any term in the sequence

$$x+1,\ x+2,\ x+3,\ \ldots,\ x+n$$

which contains q^σ, say P, a principal q-term.

On one side of P the terms are less, on the other greater than P; in lieu of any term substitute the difference between it and P, then I say that the q-index of such altered term will be the same as when it was unaltered.

For let the principal term, or the chosen principal term if there are more than one, be λq^σ, and let μq^ρ be any other term.

If $\rho < \sigma$, $\lambda q^\sigma \sim \mu q^\rho$ will obviously have ρ for its q-index; also if $\rho = \sigma$ the same will be true, *i.e.* supposing $\mu q^\rho - \lambda q^\rho$ to be positive, ρ will be its q-index: for if we write $\lambda = aq + b$ and $\mu = cq + d$, where $b < q$ and $d < q$, a and c must be equal, since otherwise between λq^ρ and μq^ρ there would be a term $(a+1)q \cdot q^\rho$ containing a higher power of q than the principal term: hence $\mu - \lambda = d - b$ and does not contain q. In like manner if $\lambda q^\rho - \mu q^\rho$ is positive, ρ is its q-index for the same reason as before.

Hence the index-sum, *quâ* any prime q, of the two sequences

$$m+1,\ m+2,\ \ldots,\ P-1,\ P+1,\ P+2,\ \ldots,\ m+n-1,\ m+n$$

is the same as the sum of the index-sums of

$$1,\ 2,\ 3,\ \ldots,\ P-m-1,$$
$$1,\ 2,\ \ldots,\ m+n-P,$$

Call the sums of these two index-sums s', then

$$s' = E\left(\frac{P-m-1}{q}\right)+E\left(\frac{P-m-1}{q^2}\right)+E\left(\frac{P-m-1}{q^3}\right)+\cdots$$
$$+E\left(\frac{m+n-P}{q}\right)+E\left(\frac{m+n-P}{q^2}\right)+E\left(\frac{m+n-P}{q^3}\right)+\cdots$$
$$= \text{ or } < E\left(\frac{n-1}{q}\right)+E\left(\frac{n-1}{q^2}\right)+E\left(\frac{n-1}{q^3}\right)+\cdots$$
$$= \text{ or } < E\left(\frac{n}{q}\right)+E\left(\frac{n}{q^2}\right)+E\left(\frac{n}{q^3}\right)+\cdots$$
$$= \text{ or } < r.$$

Hence $s' =$ or $< r$, but the original index-sum of the sequence is diminished by σ on account of P being omitted.

Hence $s-\sigma\,[=s'] =$ or $< r$.

Thus we have $s > r-1$, $s-\sigma < r+1$.

But this is not all: we may for certain relative values of m, n, and q (without regard to the situation of the principal term) establish the inequality $s-\sigma < r$.

I premise the obviously true statement that if $f+g<h$, then

$$f+E\left(\frac{f}{q}\right)+E\left(\frac{f}{q^2}\right)+\cdots+g+E\left(\frac{g}{q}\right)+E\left(\frac{g}{q^2}\right)+\cdots$$
$$<h+E\left(\frac{h}{q}\right)+E\left(\frac{h}{q^2}\right)+\cdots.$$

Let now h be the number of terms in the natural sequence from 1 to n which contain q.

Then in the given sequence the number will be

$$h+E\left(\frac{m+n}{q}\right)-E\left(\frac{m}{q}\right)-E\left(\frac{n}{q}\right), \text{ say } h+e,$$

and the sum of the number of terms divisible by q in the partial sequences on each side of P will be $h+e-1$, where $e=1$ or 0; let the respective numbers be f, g. Then $f+g=h-1+e$, where $e=0$ or 1, and, using the same notation as before,

$$s-\sigma=f+E\left(\frac{f}{q}\right)+E\left(\frac{f}{q^2}\right)+\cdots$$
$$+g+E\left(\frac{g}{q}\right)+E\left(\frac{g}{q^2}\right)+\cdots,$$

and
$$r=h+E\left(\frac{h}{q}\right)+E\left(\frac{h}{q^2}\right)+\cdots.$$

Hence if $e=0,\quad s-\sigma<r,$

if $e=1,\quad s-\sigma<r+1,$

the former inequality subsisting whenever

$$E\left(\frac{m+n}{q}\right)-E\left(\frac{m}{q}\right)-E\left(\frac{n}{q}\right)=0.$$

If for example $m=n$, $s-\sigma<r$ when

$$E\left(\frac{2n}{q}\right)-2E\left(\frac{n}{q}\right)=0.$$

which it is easily seen happens whenever $E\left(\frac{2n}{q}\right)$ is an even number.

§ 2. *Proof that* $(m+1)(m+2)\ldots(m+n)$ *when* $m>n-1$ *contains a prime not contained in* $1.2.3\ldots n$*

The universal condition independent of the relation between m, n, q, above found, viz., $s-\sigma=$ or $<r$ will be found sufficient to establish the theorem which constitutes the object of this section and which is as follows:—

"If the first term of a sequence is greater than the number of terms in it, then one term at least must be a prime or a multiple of a prime greater than that number."

When the first term exceeds by unity the number of terms, the sequence takes the form $m, m+1, m+2, \ldots, 2m-1$, and since no term in this sequence can be a multiple of n, the theorem for such case is tantamount to affirming that one term at least is a prime number which is in accord with and an easy inference from the well-known "postulate of Bertrand," that between m and $2m-2$ there must always be included some prime numbers when $m>\frac{7}{2}$.

Suppose if possible that $m+1, m+2, \ldots, m+n$ contains no other primes than such as are not greater than n, and which therefore divide some of the numbers from 1 to n.

Let q be any such prime, and P_q a principal term of the sequence

$$m+1, m+2, \ldots, m+n, \quad \text{quâ } q.$$

Then, by virtue of the proposition above established,

$$\frac{(m+1)(m+2)\ldots(m+n)}{P_q}$$

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* It will readily be seen that, if this theorem is true, for n any prime, it will be so *a fortiori* when n is a composite number.

will contain no higher power of q than does $1.2.3\ldots n$, and consequently if P be the least common multiple of the principal terms in respect to the several primes, say ν in number [unity not being reckoned one of them], none greater than n, we may infer that

$$\frac{(m+1)(m+2)\ldots(m+n)}{P}$$

will be wholly contained in, and therefore not greater than $1.2.3\ldots n$ if the sequence $m+1, m+2, \ldots, m+n$ contains no prime or multiple of a prime greater than n. To fix the ideas let us agree to consider that term in the sequence which contains the highest power of q, and is the greatest of all that do the same (if there be more than one), *the* principal q-term. The least common multiple cannot be greater than the product of the principal terms which are *distinct* from each other, and since even if they were all distinct their number cannot exceed ν (the number of primes other than unity less than $n+1$), it follows that P cannot be greater than the product of the *highest* ν terms in the given sequence. Hence we may infer that unless

$$(m+1)(m+2)\ldots(m+n-\nu)$$

is less than $1.2.3\ldots n$, some prime greater than n must divide one term at least of the sequence

$$m+1, m+2, \ldots, m+n.$$

We might go further and say that unless $1.2.3\ldots n$ is greater than

$$(m+1)(m+2)\ldots(m+n-\nu)D,$$

where $$D = \Pi q^{1+E\left(\frac{m}{q}\right)+E\left(\frac{n}{q}\right)-E\left(\frac{m+n}{q}\right)},$$

[q being made successively each of the ν primes between 2 and n inclusive and Π being used in the ordinary sense of indicating products], this same conclusion must obtain.

Conversely the theorem is true when either of these inequalities is denied. The denial of the first of them, which is sufficient for the object in view, is implied in the inequality

$$(m+1)(m+2)\ldots(m+n-\nu) > 1.2.3\ldots n,$$

which, since ν depends only on n, may be written under the form $F(m,n) > 1.2.3\ldots n$. This will be referred to hereafter, in this section, as the *fundamental inequality*.*

* The subsistence of the fundamental inequality for any given value of n implies for that value of n the truth of the theorem to be established: but the converse does not necessarily hold. The theorem may be true when the fundamental inequality is *not* satisfied.

Since $F(m, n)$ increases with m, the theorem if true for m must be true for any greater value of m, when n remains constant.

From this it will be seen at once that the theorem must be true when m has any value exceeding n^2 and $n > 7$.

For when $n = 8$ the number of primes in the range from 1 to 8 is 4 and is equal to $\frac{1}{2}n$: but as n increases the number of new primes being less than the number of odd numbers must be less than $\frac{1}{2}n$.

Hence if $n > 7$ and $m > n^2$,

$$F(m, n) > m^{n-\nu} > (n^2)^{\frac{1}{2}n} > n^n > 1.2.3 \ldots n.$$

This result enables us to prove that the theorem is true when

$$13 < n < 3000.$$

The theorem it will be borne in mind is true if some prime number occurs in the sequence $m + 1, m + 2, \ldots, m + n$, or in other words if the above sequence does not consist exclusively of composite numbers. But Dr. Glaisher has found* that the highest sequence of composite numbers within the first 9000000 contains only 153 terms, viz. the sequence 4652354 to 4652506 (both inclusive). Hence if the theorem is not true when $n < 3000$, in which case $n^2 + n < 9000000$, we must have $n =$ or < 153, and there ought to be a sequence of n composite numbers in which the first term is less than $(153)^2$ which is 23409. But the longest sequence of composite numbers under 23409 is that which extends from 19610 to 19660 containing 51 terms. the square of 51 is 2601 and the longest sequence under this number is that which extends from 1328 to 1360 comprising 33 terms. The square of 33 is 1089, the longest sequence below which is from 888 to 906 comprising 19 terms; the square of 19 is 361, the longest sequence below which stretches from 114 to 126 comprising 13 terms. Hence the theorem is true for all values of n not greater than 3,000 and not less than 13.

It is easy to show that the theorem is true for all values of n not greater than 13.

1°. Suppose $n = 13$, which gives $\nu = 6$.

The theorem must be true when m is taken so great that

$$(m + 1)(m + 2)(m + 3)(m + 4)(m + 5)(m + 6)(m + 7)$$
$$> 1.2.3.4.5.6.7.8.9.10.11.12.13,$$

which is easily seen to be satisfied when $m =$ or > 100.

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* See table at end of this section.

But there is no sequence of 13 composite numbers till we come to the sequence 114–126, so that when $m < 100$ the theorem must be true as well as when $m =$ or > 100.

2°. Suppose $n = 11$, for which value of n, $\nu = 5$.
The theorem is true if

$$(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)$$

$$> 1.2.3.4.5.6.7.8.9.10.11,$$

which is obviously satisfied as before when $m = 100$, but there is no sequence of 11 which precedes the sequence before named from 114 to 126. Hence the theorem is true generally for $n = 11$.

When $n = 7$, $\nu = 4$ and the theorem is true for all values of m which make

$$(m+1)(m+2)(m+3) > 1.2.3.4.5.6.7, \text{ i.e. } > 5040,$$

which is obviously the case if $m =$ or > 20, but there is no sequence of 7 composite numbers till we come to 89–97. Hence the theorem is proved for $n = 7$.

When $n = 5$, $\nu = 3$ and the condition of the theorem is satisfied if

$$(m+1)(m+2) > 2.3.4.5, \text{ i.e. } > 120,$$

as is the case if $m =$ or > 10, but the first composite sequence of 5 terms is 24 to 28. In like manner when $n = 3$, $\nu = 2$ and the theorem is true when $m + 1 =$ or $> 1.2.3$, *i.e.* $m =$ or > 5, but 8, 9, 10 is the first composite sequence of 3 terms. Similarly, when $n = 2$, $\nu = 1$ and the condition $m + 1 = > 2$ is necessarily satisfied since $m =$ or $> n$ by hypothesis.

Finally, the theorem is obviously true when $n = 1$, because $m + 1$ whatever m may be, contains a factor greater than 1.

Being true for the prime numbers not exceeding 13, the slightest consideration will serve to prove that, as previously remarked in a footnote, it must be true *à fortiori* for all the composite numbers between them. Hence the theorem is verified for all values of n not greater than 3000, and it only remains to establish it for values of n exceeding that limit.

To prove it for this case we must begin with finding a superior limit to ν, when $n > 3000$, under the convenient form of a multiple of $\frac{n}{\log n}$.

If we multiply together the first 9 prime numbers from 2 to 23 and divide their product by that of the natural numbers up to 9 increased in the ratio of 1 to 2^9, the quotient will be found to exceed unity; and since the following primes are all more than twice the corresponding natural numbers, if we denote by $p_1, p_2, p_3, \ldots$, the prime numbers 2, 3, 5, ..., we must have

$$p_1 . p_2 . p_3 . \ldots . p_\nu > 2^\nu (1.2.3\ldots\nu),$$

[provided that $\nu > 22$, as is the case if $n =$ or > 89],

or $$\log (1.2.3\ldots\nu) + (\log 2)\,\nu < \log (p_1 . p_2 . p_3 . \ldots . p_\nu).$$

But by Stirling's theorem (Serret, *Cours d'Alg. Sup.*, Ed. 4, Vol. II., p. 226),

$$\nu \log \nu - \nu - \tfrac{1}{2} \log \nu + \tfrac{1}{2} \log 2\pi < \log (1.2.3\ldots\nu),$$

and by Tschebyscheff's theorem (Serret, Vol. II., p. 236),*

$\log (p_1 . p_2 . p_3 \ldots p_\nu) < n'$, where

$$n' = \tfrac{6}{5} A n + \frac{5}{4 \log 6} (\log n)^2 + \tfrac{5}{2} \log n + 2, \text{ and } A = {\cdot}921292\ldots,$$

Hence

$$(\log \nu)(\nu - \tfrac{1}{2}) - (1 - \log 2)(\nu - \tfrac{1}{2}) + (\tfrac{1}{2} \log 2\pi - \tfrac{1}{2} \log \tfrac{1}{2} e) < n',$$

and *à fortiori*

$$\log (\nu - \tfrac{1}{2})(\nu - \tfrac{1}{2}) - (\log \tfrac{1}{2} e)(\nu - \tfrac{1}{2}) < n',$$

or $$\frac{2}{e} (\nu - \tfrac{1}{2}) \log \left\{ \frac{2}{e} (\nu - \tfrac{1}{2}) \right\} < \frac{2}{e} n'.$$

Hence, if we write $\mu \log \mu = \dfrac{2}{e} n' = n_1$

we shall have $$\nu - \tfrac{1}{2} < \tfrac{1}{2} e \mu.$$

But $$\mu = \frac{n_1}{\log \mu},$$

and therefore

$$\log \mu = \log n_1 - \log \log \mu = \log n_1 - \log (\log n_1 - \log \log \mu)$$
$$> \log n_1 - \log \log n_1.$$

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* For greater simplicity I have left out the term $- A n^{\frac{1}{2}}$, and thereby *increased* the superior limit.

Hence $\mu < \dfrac{n_1}{\log n_1 - \log\log n_1}$

$$< \frac{2}{e}\,\frac{n'}{\log n' - \log\log n' + \log\frac{2}{e}}$$

and $\nu < \frac{1}{2} + \dfrac{n'}{\log n' - \log\log n' - (1-\log 2)}$.*

Hence, observing that $\dfrac{1}{u}$, $\dfrac{\log u}{u}$, $\dfrac{(\log u)^2}{u}$, $\dfrac{\log\log u}{\log u}$ *all* decrease as the denominators increase [provided as regards the second of these fractions that $u > e$, as regards the third that $u > e^2$, and as regards the fourth that $u > e^e$], we may find a superior limit to ν in the case before us, where $n > 3000$, by writing in the numerator of $\nu - \frac{1}{2}$,

$$\frac{(\log 3000)^2}{3000}n,\quad \frac{\log 3000}{3000}n,\quad \frac{2}{3000}n,$$

for $(\log n)^2$, $\log n$, 2,

and in its denominator, first, $\log n - \log\log n$ for $\log n' - \log\log n'$, and then

$$\frac{\log\log 3000}{\log 3000}\log n \text{ and } \frac{1-\log 2}{\log 3000}\log n,$$

for $\log\log n$ and $1-\log 2$ respectively.

Making the calculations it will be found that we shall get

$$\nu - \tfrac{1}{2} < 1{\cdot}606\,\frac{n}{\log n}.$$

With the aid of this limit it will now be easy to prove the truth of the theorem when $n =$ or > 3000.

Let us suppose $n =$ or > 3000.

1.° Suppose $m < 2n$, then $m + n > \frac{3}{2}m$ and the theorem will be proved for this case, if it can be shown that in the range of numbers from m to $\frac{3}{2}m$, there is at least one prime number when $m =$ or > 3000.

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* From this it will be seen that the asymptotic ratio of ν to $\dfrac{n}{\log n}$ is less than the asymptotic ratio which any superior limit to the sum of the logarithms of the primes not exceeding n bears to n: this perhaps is a new result, at all events it is not to be found in Serret nor indeed is it wanted for Tschebyscheff's proof of the famous postulate which Serret has so lucidly expounded. The correlative theorem that the asymptotic ratio of ν to $\dfrac{n}{\log n}$ is always *greater* than the asymptotic ratio which any *inferior* limit to the sum aforesaid bears to n is of course an obvious and familiar fact.

This will be the case (Serret, Vol. II., p. 239), if (on that supposition) $\frac{5}{6} \cdot \frac{3}{2}n - n$, *i.e.*, if

$$\frac{n}{4} > 2\sqrt{(\tfrac{3}{2}x)} + \frac{25\,(\log \frac{3}{2}n)^2}{16A \log 6} + \frac{125}{24A}(\log \tfrac{3}{2}n) + \frac{25}{6A},$$

where $A = \cdot 92129202 \ldots$.

But when $n = 3000$, it will be found that the terms on the second side of the inequality are respectively less than

$$134\cdot1641, \quad 66\cdot9773, \quad 47\cdot5546, \quad 4\cdot5227,$$

whose sum is less than 750.

Hence, the inequality is satisfied, and accordingly the theorem is true when $m < 2n$ and n is equal to or *greater* than 3000; for when n satisfies that condition the derivative in respect to n of the right-hand side of the above inequality will be always less than $\frac{1}{4}$.

2.° Suppose $m =$ or $> 2n$, then it is only necessary to prove that
$$\log (2n+1)(2n+2)\ldots(3n-\nu) > \log (1.2.3\ldots n),$$
or, what is the same thing, that
$$\log \{1.2.3.4\ldots(3n-\nu)\} > \log (1.2.3\ldots n) + \log (1.2.3\ldots 2n),$$
ν being the number of primes not greater than n, and n being at least 3000.

Call the two sides of the inequality P and Q.

Then (Serret, Vol. II., p. 226).

$$P > \log\sqrt{(2\pi)} + (3n-\nu)\log(3n-\nu) - (3n-\nu) - \tfrac{1}{2}\log(3n-\nu)$$

$$> \log\sqrt{(2\pi)} + (3n-\nu)\log 3n + (3n-\nu)\log\left(1 - \frac{\nu}{3n}\right) - 3n + \nu - \tfrac{1}{2}\log 3n$$

$$> \log\sqrt{(2\pi)} + 3(\log n)\,n + (3\log 3 - 3)\,n - (\log n)\,\nu$$
$$+ (1 - \log 3)\,\nu - \tfrac{1}{2}\log 3 - \tfrac{1}{2}\log n - \nu,$$

for $-(3n-\nu)\log\left(1 - \dfrac{\nu}{3n}\right)$

$$= \nu\left\{1 - \tfrac{1}{2}\left(\frac{\nu}{3n}\right) - \tfrac{1}{6}\left(\frac{\nu}{3n}\right)^2 - \tfrac{1}{12}\left(\frac{\nu}{3n}\right)^3 - \ldots\right\} < \nu.$$

On the other hand,

$$Q < \log\sqrt{(2\pi)} + n\log n - n + \tfrac{1}{2}\log n + \tfrac{1}{12}$$
$$+ \log\sqrt{(2\pi)} + 2n\log 2n - 2n + \tfrac{1}{2}\log 2n + \tfrac{1}{12}$$
$$< \{2\log\sqrt{(2\pi)} + \tfrac{1}{2}\log 2 + \tfrac{1}{6}\} + 3(\log n)\,n + (2\log 2 - 3)\,n + \log n.$$

Hence $P - Q > (3\log 3 - 2\log 2)\,n - (\log n)\,\nu$

$$- \tfrac{3}{2}\log n - (\log 3)\,\nu - \{\tfrac{1}{2}\log(12\pi) + \tfrac{1}{6}\}$$

$$> (3\log 3 - 2\log 2)\,n - \log n\,(\nu - \tfrac{1}{2})$$

$$- 2\log n - \log 3\,(\nu - \tfrac{1}{2}) - \{\tfrac{1}{2}\log(36\pi) + \tfrac{1}{6}\}$$

where $\nu - \frac{1}{2} < 1{\cdot}606\,\dfrac{n}{\log n}$.

But $3\log 3 - 2\log 2 = 1{\cdot}9095415 > 1{\cdot}909$.

Hence*

$$P - Q > ({\cdot}303)\,n - (1{\cdot}606\log 3)\frac{n}{\log n} - 2\log n - \{\tfrac{1}{2}\log(36\pi) + \tfrac{1}{6}\},$$

say $P - Q > f(n) > 0$ when $n = 3000$.

Also the derivative with respect to n of $(\log n) f(n)$ being

$$({\cdot}303)(1 + \log n) - 1{\cdot}606\log 3 - \frac{4\log n}{n} - \frac{\frac{1}{2}\log(36\pi) + \frac{1}{6}}{n},$$

$P - Q$ will increase as n increases and will remain positive for all values of n superior to 3000.

Hence the theorem is true, whatever m may be, when $n =$ or > 3000, and since it has been proved previously for the case of $n < 3000$, it is true universally.

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* It will now be seen why I take separately the two cases of m greater and m less than $2n$. If we were to take *simpliciter* $m =$ or $> n$ and were to attempt to prove $\log\{1.2.3\ldots(2n-\nu)\} > 2\log(1.2.3\ldots n)$ the inferior limit to the difference between these two quantities would then have for its principal term, not $(3\log 3 - 2\log 2 - 1{\cdot}606)\,n$ but $(2\log 2 - 1{\cdot}606)\,n$, which would be *negative*.

Of course there is no special reason except of convenience (in dealing with an integer instead of a fraction) for making $2n$ the dividing point between the two suppositions separately considered in the text; κn where κ as far as regards the second inequality does not fall short of some certain limit, would have served as well: this inferior limit to κ would be some quantity a little greater (how much exactly would have to be found by trial) than the quantity θ which makes $\theta\log\theta - (\theta - 1)\log(\theta - 1)$ equal to the coefficient of $\dfrac{n}{\log n}$ in the superior limit to ν. As regards the first inequality κ would have to be a quantity somewhat less (how much less to be found by trial) than the quantity η which makes $\dfrac{\eta + 1}{\eta} = \frac{6}{5}$, *i.e.* $\eta = 5$. This is on the supposition made throughout of using Tschebyscheff's own limits, but if we use the more general, but less compact, limits indicated in my paper in Vol. IV. of the *American Journal of Mathematics*, any fraction not less than $\frac{6}{5}$ and not so great as $\frac{59596}{51073}$ would take the place of $\frac{6}{5}$, and the extreme value of η would be $\frac{51073}{8523}$, which is a trifle under 6. By a judicious choice of the value given to κ, a value of n could be found considerably less than 3000, which would satisfy both inequalities, and this in the absence of Dr. Glaisher's table would have been a matter of some practical importance, but is of next to none when we have that table to draw upon. How low down in the scale of number n may be taken without interruption of the existence of the fundamental inequality for the minimum value of n in the case treated of in this section, it has not been necessary for the purpose in hand to ascertain. That it holds good for all values of n above a certain limit follows from the fact that $2\log 2$ is greater than the coefficient of the leading term in the superior functional limit to the sum of the logarithms of the primes not greater than n.

I subjoin the valuable table, kindly communicated to me by Dr. Glaisher, referred to in the text above.

Table of Increasing Sequences of Composite Numbers interposed between Consecutive Primes included in the first nine million numbers.

Limits to Sequence.			Number of terms.
7	to	11	3
23	„	29	5
89	„	97	7
113	„	127	13
523	„	541	17
887	„	907	19
1129	„	1151	21
1327	„	1361	33
9551	„	9587	35
15683	„	15727	43
19609	„	19661	51
31397	„	31469	71
155921	„	156007	85
373261	„	373373	111
492113	„	492227	113
1349533	„	1349651	117
1357201	„	1357333	131
2010733	„	2010881	147
4652353	„	4652507	153

The table is to be understood as follows. The lowest sequence of as many as 3 consecutive composite numbers is that included between 7 and 11: the lowest of as many as 5 is that included between 23 and 29, of as many as 7 that included between 89 and 97; between 13 and 17 there is a break—this indicates that the lowest sequence of as many as 15, or as many as 17 first occurs in the sequence of 17 interposed between 523, 541. Similarly the break between 21 and 33 indicates that the lowest sequence containing 23 or 25 or 27 or 29 or 31 or 33 terms first occurs in the sequence of 33 composite numbers interposed between the primes 1327, 1361.

It is also necessary to add that in the first nine millions numbers there is no succession of *more* than 153 consecutive composite numbers.

§ 3. *Relating to irreducible arithmetical series in general.**

Let P be a principal term quâ q in any *irreducible* arithmetical series whose common difference is i, N any other term greater or less than P, and D their difference. If q is not prime to i, no term in the series will be divisible by q.

Just as in the case of a natural seqence when there is only one principal term in the series it may be shown that the index of D quâ q will be the same as that of N; when there is more than one principal term it appears by the same reasoning as before that the index of N cannot be greater than that of D: [it will not now necessarily be equal unless q is greater than the common difference i].

The index-sum quâ q is zero when q has a common measure with i, and we may therefore consider only the case where q is relatively prime to i; on this supposition, by virtue of what has been stated above, the index-sum quâ q of the series whose first term is $m+i$, and number of terms n, will be equal to or less than

$$E\left(\frac{P-m-i}{iq}\right)+E\left(\frac{P-m-i}{iq^2}\right)+E\left(\frac{P-m-i}{iq^3}\right)+\cdots$$
$$+E\left(\frac{m+ni-P}{iq}\right)+E\left(\frac{m+ni-P}{iq^2}\right)+E\left(\frac{m+ni-P}{iq^3}\right)+\cdots;$$

and therefore *à fortiori*

$$< \text{ or } = E\left(\frac{(n-1)\,i}{iq}\right)+E\left(\frac{(n-1)\,i}{iq^2}\right)+E\left(\frac{(n-1)\,i}{iq^3}\right)+\cdots$$
$$< \text{ or } = E\left(\frac{n}{q}\right)+E\left(\frac{n}{q^2}\right)+E\left(\frac{n}{q^3}\right)+\cdots,$$

i.e. not greater than the index-sum of 2, 3, ..., n quâ q.

Consequently, by the same reasoning as that employed in the last section, the theorem now to be proved, viz. that if m [prime to i] $=> n$, then $(m+i)(m+2i)\ldots(m+ni)$ must contain some one or more prime numbers greater than n, must be true whenever

$$(m+i)\,(m+2i)\,(m+3i)\ldots(m+(n-\nu_1)\,i) > 1.2.3\ldots n \ \ldots(\Theta)\dagger$$

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* An irreducible arithmetical series is one whose terms are prime to their common difference.

† If it had been necessary the condition in the text might have been stated in the more stringent form that *some aliquot part* of the factorial of n (viz. this factorial divested of all powers of prime numbers contained in i) would have to be greater than

$$(m+i)\,(m+2i)\ldots\{m+(n-\nu_1)\,i\}$$

if the theorem were not true for any specified values of m, n, i.

It will be noticed that when i is relatively prime to n, ν_1 is less than ν so that $n-\nu_1 > n-\nu$: some use will be made of the formula in the text when dealing with certain small values of n and $m-n$ towards the end of the section.

where ν_1 is the number of prime numbers not exceeding n, and not contained in i, and *à fortiori* when for ν_1, we substitute, as for the present we shall do, ν the *entire* number of primes not greater than n. This I term the *fundamental inequality* for the general case now under consideration.

Suppose $n =$ or > 3000. The logarithm of the first side of the fundamental inequality when we write ν for ν_1 is obviously greater than the i^{th} part of the logarithm of

$$(m+i)(m+i+1)(m+i+2)\ldots\{m+(n-\nu)i\},$$

and the inequality (subject to certain suppositions) to be established will be satisfied, if on the same suppositions,

$$\frac{1}{i}\log[1.2.3\ldots\{m+(n-\nu)i\}] > \left(\frac{1}{i}+1\right)\log(1.2.3\ldots n).$$

Suppose $m = n$, and make

$$\log[1.2.3\ldots\{(i+n)n - i\nu\}] = T,$$

$$(i+1)\log(1.2.3\ldots n) = U,$$

$$F(n, i) = T - U.$$

Then $T > \log(2\pi) + \{(i+1)n - i\nu\}\log\{(i+1)n - i\nu\}$

$$- \{(i+1)n - i\nu\} - \tfrac{1}{2}\log\{(i+1)n - i\nu\},$$

$$U < (i+1)\log\sqrt{(2\pi)} + (i+1)n\log n - (i+1)n + \tfrac{1}{2}(i+1)\log n + \tfrac{1}{12}(i+1).$$

Hence $F(n, i) > -i\log\sqrt{(2\pi)} + \{(i+1)n - i\nu\}\log\{(i+1)n\}$

$$+ \{(i+1)n - i\nu\}\log\left\{1 - \frac{i\nu}{(i+1)n}\right\}$$

$$+ i\nu - (i+1)n\log n - \tfrac{1}{2}\log\{(i+1)n - i\nu\} - \tfrac{1}{2}(i+1)\log n - \tfrac{1}{12}(i+1)$$

$$> \{(i+1)\log(i+1)\}n - i\{\log(i+1)n\}\nu - \tfrac{1}{2}\log\{(i+1)n\} - \tfrac{1}{2}(i+1)\log n$$

$$- \tfrac{1}{2}i\log(2\pi) - \tfrac{1}{12}(i+1),$$

i.e. $> \{(i+1)\log(i+1)\}n - i\{\log(i+1)n\}\nu - \tfrac{1}{2}(i+2)\log n - \tfrac{1}{2}\log(i+1)$

$$- \tfrac{1}{2}i\log(2\pi) - \tfrac{1}{12}(i+1),$$

so that when $n > 3000$ and consequently $\nu < \tfrac{1}{2} + (1{\cdot}606)\dfrac{n}{\log n}$,

the inequality to be established will be true *à fortiori* if

$$F(n,i) > \left\{(i+1)\log(i+1) - (1{\cdot}606)i\left[1 + \frac{\log(i+1)}{\log n}\right]\right\}n - (i+1)\log n$$

$$- \left[\tfrac{1}{2}(i+1)\log(i+1) + \tfrac{1}{2}\{i\log(2\pi)\} + \tfrac{1}{12}(i+1)\right]\ldots(II).$$

When $i=1$ or 2 or 3 the coefficient of n is negative; consequently the limit to ν before found is no longer applicable to bring out the desired result.

The case of $i=1$ has been already disposed of; that of $i=2$ may be disposed of, as I shall show, in a similar manner; when $i=3$, I shall raise the limit n from 3000 to 8100 of which the logarithm is so near to 9 that it may, for the purpose of the proof in hand, be regarded as equal to 9 without introducing any error in the inequality to be established, as the error involved will only affect the result in a figure beyond the 4th or 5th place of decimals, whereas the inequality in question depends on figures in the first decimal place. When this is done the theorem will be in effect demonstrated for the case of $i=3$ and $n>8100$. For all values of n not greater than 8100 I shall be able to show that the fundamental inequality (Θ) is satisfied by employing the actual value of ν_1 or ν instead of a limiting value of the latter.

Thus the fundamental inequality will be shown to subsist for all values of n when $i=3$ and $m=n$, and *à fortiori* therefore for all values of m and i *not less* than n and 3 respectively.

Case of $i=2$.

Suppose $n=>3000$, and take separately the cases $m<=2n$, $m>2n$.

1°. Let m be not greater than $2n$ so that $m+2n$ is greater than $2m-1$.

By hypothesis m must be odd, and by Bertrand's Postulate

$$m+2,\ m+3,\ m+4,\ \ldots,\ 2m,$$

and therefore $m+2,\ m+4,\ m+6,\ \ldots,\ (2m-1)$

(seeing that the interpolated terms are all even) must contain a prime, and thus the first case is disposed of.

2°. Since the fundamental inequality has been shown to be satisfied when $m>\frac{3}{2}n$ it will be true *à fortiori* when $m>2n$.

Hence the theorem is established for $i=2$ when $n>3000$. Finally as regards values of n inferior to 3000, the reasoning employed for the case of $i=1$ applies *à fortiori* to the case of $i=2$.

To see this let us recall the first step of the reasoning applicable to the supposition of $i=1$.

Because in the first nine million numbers there is no sequence of 3000 composite numbers, from Dr. Glaisher's Table of Sequences (taken in conjunction with the fact that when $m > n^2$, the theorem has been proved to be true whatever n may be), we were able to infer that it must be true when n does not exceed 153: in the present case, if the theorem were not true when $3000 > n > 153$, there would be a sequence of 153 composite odd numbers and therefore of over 305 composite consecutive numbers in the first 9000000 numbers, whereas there are not more than 153, and so we may proceed step by step till we arrive at the conclusion that the theorem must be true when $n > 13$; and when $n = 13, 11, 7, 5, 3, 2, 1$ a like method of disproof (but briefer) will apply as for the case of $i = 1$.

Case of $i = > 3$.

Let $n = > 8100$. Then we may without ultimate error write

$$\nu - \tfrac{1}{2} < \frac{1{\cdot}1056 + \frac{5}{4\log 6}\frac{81}{8100} + \frac{5}{2}\frac{9}{8100} + \frac{2}{8100}}{1 - \frac{\log 9}{9} - \frac{1-\log 2}{9}} \frac{n}{\log n} < 1{\cdot}546 \frac{n}{\log n},$$

and accordingly

$$F(n, 3) > \left\{4\log 4 - (3 \times 1{\cdot}546)\left(1 + \frac{\log 4}{9}\right)\right\} n$$

$$- 4\log n - (2\log 4 + \tfrac{3}{2}\log 2\pi + \tfrac{1}{3})$$

and $F(8100, 3) > (5.545 - 5{\cdot}352)(8100) - 36 - 5{\cdot}863 > 0.$

Hence the Fundamental Inequality is satisfied when $n = > 8100$.

To prove that it is satisfied for values inferior to 8100, observe that by virtue of the formula (H) it will be so, *ex abundantiâ*, for all values of n not less than $'n$ and not greater than n', provided that, calling n'_ν the number of primes not exceeding n',

$$(5{\cdot}545)\,'n - 3\log(4n')\,n'_\nu - \tfrac{5}{2}\log n' - C > 0,$$

where $$C = \tfrac{1}{3} + \log 2 + \tfrac{3}{2}\log(2\pi) = 3{\cdot}783.$$

On trial it will be found that the above inequality is satisfied when we successively substitute for $'n$, n', and for n'_ν (found from any Table for the enumeration of primes) the values given in the annexed table:

n'	n'_ν	$'n$
8100	1018	5725
5724	753	4096
4095	564	2967
2966	427	2172
2171	326	1604
1603	252	1200
1199	196	903
902	154	687
686	124	535
534	99	415
414	80	325
324	66	260
259	55	210
209	46	171
170	39	141
140	34	111
110	29	99
98	25	84
83	23	76
75	21	68
67	19	62
61	18	57
56	16	50
49	15	46
45	14	42
41	13	39
38	12	36
35	11	32
31	11	31
30	10	30
29	10	29

The fundamental theorem is therefore established when $i > 2$ for all values of n down to 29 inclusive.

It remains to consider the case where n is any prime number less than 29.

Calling μ the difference between n and the number of primes (exclusive of 1) not greater than n, to

$$n = 2,\ 3,\ 11,\ 17,\ 23$$

will correspond

$$\mu = 1,\ 1,\ 6\ ,\ 10,\ 14$$

and for each combination of these corresponding numbers it will be found that

$$1.2.3\ldots n = \text{or} < (n+3)(n+6)\ldots(n+3\mu).$$

Hence the theorem is proved for these values of n, whatever n may be, when $i = > 3$. To

$$n = 13, \ n = 19$$

corresponds

$$\mu = 7 \ , \ \mu = 11,$$

and for these combinations of n and μ it will be found that

$$1.2.3\ldots n < (n+4)(n+7)\ldots(n+1+3\mu),$$

so that the theorem is true for

$$n = 13, 19,$$

except in the case where

$$m = 13, 19.$$

That it is true in these excepted cases follows from inspection of the series,

16, 19, 22, 25, &c.,

22, 25, 28, 31, &c.,

where $19 > 13$, $31 > 19$: or it might be proved, but more cumbrously, by the same method as that applied below to the only two values of n remaining to be considered, viz.

$$n = 5, \ n = 7,$$

for which we have respectively

$$\mu = 2, \ \mu = 3.$$

If $n = 5$ and i has no common measure with 2.3.4.5, i must be not less than 7, but $1.2.3.4.5 < 12.19$.

On the other hand, if i has a common measure with 2.3.4 5, then what we have called ν_1, in formula (Θ), is less than ν, so that $n - \nu_1 > 2$, but

$$1.2.3.4.5 < 8.11.14.$$

These two inequalities combined serve to prove that, whatever i may be, the inequality (Θ) is satisfied, and the theorem is consequently proved for $n = 5$.

So again, when $n = 7$, if i has no common measure with 2.3.4.5.6.7 it must be 11 at least. In that case the inequality $2.3.4.5.6.7 < 18.29.40$, and in the contrary case the inequality $2.3.4.5.6.7 < 10.13.16.19$ serves to prove the theorem.

When $n=1$ the truth of the theorem is obvious: hence combining the results obtained in this and the preceding section, it will be seen we have proved that whatever n and whatever i may be, provided that m is relatively prime to i and not less than n, the product

$$(m+i)(m+2i)\dots(m+ni)$$

must contain some prime number by which $2.3\dots n$ is not divisible, and the wearisome proof is thus brought to a close. It will not surprise the author of it, if his work should sooner or later be superseded by one of a less piece-meal character—but he has sought in vain for any more compendious proof. He has not thought it necessary to produce the figures or refer in detail to the calculations giving the numerical results inserted in various places in the text: had he done so the number of pages, already exceeding what he had any previous idea of, would probably have been more than doubled.

New College, Oxford
June 6, 1891.

*End of Part I.**

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* The author was wandering in an endless maze in his attempts at a general proof of his theorem, until in an auspicious hour when taking a walk on the Banbury road (which leads out of Oxford) the Law of Ademption flashed upon his brain: meaning thereby the law (the nerve, so to say, of the preceding investigation) that *if all the terms of a natural arithmetical series be increased by the same quantity so as to form a second such series, no prime number can enter in a higher power as a factor of the product of the terms of this latter series, when a suitable term has been* taken away *from it, than the highest power in which it enters as a factor into the product of the terms of the original series.*

In Part II I shall be able to apply the same method to demonstrate a theorem showing that it is always possible to split up an infinite arithmetical series, if not in the general case, at least for certain values of the common difference, into an infinite number of successive finite and determinable segments such that one or more primes shall be found in each such segment: a theorem which is, so to say, Dirichlet's theorem on arithmetical progressions cut up into slices.

The whole matter is thus made to rest on an elementary fundamental *equality* (Tschebyscheff's) which, with the aid of an application of Stirling's theorem, leads (as the former has so admirably shown) *inter alia* to a superior limit to the sum of the logarithms of the primes not exceeding a given number, from which as has been seen in §2, a superior limit may be deduced to the number of such primes. With the aid of this last limit together with an elementary fundamental *inequality* and a *renewed* application of Stirling's theorem, all my results are made to flow. Thus a theorem of pure form is brought to depend on considerations of greater and less, or as we may express it, Quality is made to stoop its neck to the levelling yoke of Quantity.

Long and vain were my previous efforts to make the desired results hinge upon the properties of transposed Eratosthenes' scales: now we may hope to reverse the process and compel these scales to reveal the secret of their laws under the new light shed upon them by the successful application of the Quantitative method.
