ON THE FORM OF CLOSED CURVES OF THE THIRD CLASS.

By *W. Burnside.*

THE number of real tangents that can be drawn to an algebraical curve from a given point of its plane depends on the part of the plane in which the point is situated, and the whole plane can be divided into a finite number of regions in each of which the number of real tangents that can be drawn from a point is constant. As the number in question can only change by pairs of tangents passing from being real to being imaginary, the lines bounding the different regions must clearly be the curve itself and the tangents at the inflections and at those cusps at which both branches lie on one side of the tangent. Cusps of this kind can clearly not occur on curves of the third class, as from a suitably chosen point in the neighbourhood of such a cusp four real tangents can always be drawn to the curve; and for the same reason it is clear that a double point with real and different tangents cannot occur either.

For curves of the third class without infinite branches (*i.e.* closed curves) it may be shown very simply that inflections also cannot occur. For if there is one inflection there must be another in order that the curve may re-enter; but from the point of intersection of the tangents at the inflections the equivalent of four real tangents can be drawn to the curve, which is impossible. A closed curve of the third class must therefore consist of one or more separate non-intersecting portions without inflections and with no other singularities than ordinary cusps.

First, suppose it consists of a single portion. This cannot be a convex oval, for such a curve would divide the plane into the regions from which two and no tangents respectively could be drawn to the curve. It must therefore be a closed concave curve presenting three cusps at least (as it is impossible to draw such a curve with two cusps only and no inflections). Also it cannot have more than three cusps, for it is at once obvious from a figure that the number of tangents that can be drawn from any point on the inside of such a curve is equal to the number of cusps. Hence, one possible form is a single closed three-cusped curve dividing the plane into two regions from which three tangents and one respectively can be drawn to the curve.

If the curve consists of two separate portions, one at least must be such as that just found, as otherwise they would divide the plane into regions from each of which an even number of tangents could be drawn. Also from points inside the cusped portion it must be impossible to draw real tangents to the other portion; and this must therefore be an oval convex curve enclosing the former.

Finally, the curve cannot consist of more than two portions, as any additional closed curve added to the previous two would make it possible to draw at least five real tangents from suitably chosen points. This second possible form, consisting of a three-cusped portion surrounded by an oval, will divide the plane into three regions, from which three, one and three real tangents respectively can be drawn.

If the curve is symmetrical, with the same lines of symmetry as an equilateral triangle, its line equation can be written in the form

$$l^3 + m^3 + n^3 - 3klmn = 0.$$

The tangents at the cusps are evidently from symmetry $(0, 1, -1)$, $(1, 0, -1)$, $(1, -1, 0)$ and therefore the coordinates of the cusps are $(k, 1, 1)$, $(1, k, 1)$, $(1, 1, k)$.

When $k = -2$ these lie on the line at infinity and the curve is passing from a form with infinite branches to a closed form. When $k = 0$ the curve is the three-cusped hypocycloid; and when $k = 1$ the three-cusped branch has closed up to a point while the line at infinity appears as part of the curve and gives the first indication of the oval surrounding the three-cusped branch. As k increases from the value unity the three-cusped branch opens out again with the cusps now pointing in the opposite direction to that they did before, while the encircling oval grows smaller, always, however, remaining outside the triangle; the three-cusped branch being entirely inside. As k increases without limits the two branches approach the triangle continually from within and without respectively.

ALGEBRAICAL NOTES.

By *W. Burnside.*

I. *On the Jacobian of two quadratics.*

It is a known theorem that if u and v are quadratic functions of x, then

$$\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)^2 = Au^2 + Buv + Cv^2,$$