NOTE ON THE SUM OF FUNCTIONS OF QUANTITIES WHICH ARE IN ARITHMETICAL PROGRESSION.

By *H. W. Richmond*, King's College, Cambridge.

In No. 11, Vol. xx. of the *Messenger of Mathematics*, Dr. Glaisher points out that if the sum of any power of the series of numbers 2, 4, 6, ..., n be expressed as a function of n, the sum of the same power of the series 1, 3, 5, ..., n will be the same function of n, save for a numerical constant. More generally, if

$$\phi(0)+\phi(q)+\phi(2q)+\ldots+\phi(n)$$

be expressed as a function of n, then the sum of the series

$$\phi(r)+\phi(r+q)+\phi(r+2q)+\ldots+\phi(n)$$

will be the same function of n increased by a numerical constant. I wish to discuss some properties of this constant numerical quantity.

1. If $\phi(n)$ be a rational integral function of n, the series

$$\phi(0)+\phi(2)+\phi(4)+\ldots+\phi(n)=\frac{E^{n+2}-1}{E^2-1}\phi(0).$$

Now if n be any integer, odd or even, the operator $\dfrac{E^{n+2}-1}{E^2-1}$ may be expanded in ascending positive powers of Δ, and therefore $\dfrac{E^{n+2}-1}{E^2-1}\phi(n)$ can be expressed as a rational integral finite function of n, $=\psi(n)$. Thus

$$\phi(0)+\phi(2)+\phi(4)+\ldots+\phi(n)=\psi(n).$$

Then

$$\phi(1)+\phi(3)+\phi(5)+\ldots+\phi(n)=\frac{E^{n+2}-E}{E^2-1}\phi(0)$$

$$=\frac{E^{n+2}-1}{E^2-1}\phi(0)-\frac{1}{E+1}\phi(0)$$

$$=\psi(n)-\lambda,$$

where λ is a constant, viz. $\dfrac{1}{E+1}\phi(0)$.

Should $\phi(n)$ contain only even powers of n,

$$f(E)\,\phi(0) = f(E^{-1})\,\phi(0).$$

Hence $\lambda = \dfrac{1}{E^{-1}+1}\phi(0) = \dfrac{E}{E+1}\phi(0) = \phi(0) - \lambda.$

That is $\lambda = \frac{1}{2}\phi(0).$

Hence if $\phi(n)$ denote any rational integral function of n^2, and if

$$\phi(0) + \phi(2) + \phi(4) + \ldots + \phi(n) = \psi(n),$$

then $\phi(1) + \phi(3) + \phi(5) + \ldots + \phi(n) = \psi(n) - \frac{1}{2}\phi(0).$

2. When $\phi(n)$ contains uneven powers of n, the value of λ is less simple.

$$\lambda = \frac{1}{1+E}\phi(0) = \frac{1}{1+e^D}\phi(0)$$

$$= \left[\tfrac{1}{2} - \frac{2^2-1}{2!}B_1D + \frac{2^4-1}{4!}B_2D^3 - \frac{2^6-1}{6!}B_3D^5 \ldots\right]\phi(0),$$

where $B_1, B_2, B_3, \ldots$ are Bernoulli's numbers.

Hence, if $\phi(n) = a_0 + a_1n + a_2n^2 + a_3n^3 + \ldots,$

$$\lambda = \frac{a_0}{2} - \frac{2^2-1}{2}B_1a_1 + \frac{2^4-1}{4}B_2a_3 - \frac{2^6-1}{6}B_3a_5 \ldots.$$

and in particular, if

$$0^{2p} + 2^{2p} + 4^{2p} + \ldots + n^{2p} = \psi(n),$$

then $1^{2p} + 3^{2p} + 5^{2p} + \ldots + n^{2p} = \psi(n),$

But if

$$0^{2p-1} + 2^{2p-1} + 4^{2p-1} + \ldots + n^{2p-1} = \psi(n),$$

then $1^{2p-1} + 3^{2p-1} + 5^{2p-1} + \ldots + n^{2p-1} = \psi(n) - (-1)^p \dfrac{2^{2p}-1}{2p}B_p.$

3. In the same way, $\phi(n)$ being a rational integral function of n, the quantities

$$\frac{E^{n+3}-1}{E^3-1}\phi(0),\ \frac{E^{n+4}-1}{E^4-1}\phi(0),\ \ldots,\ \frac{E^{n+q}-1}{E^q-1}\phi(0),$$

are all rational integral functions of n, which I denote by

$$\psi_3(n),\ \psi_4(n),\ \ldots,\ \psi_q(n),$$

respectively.

Then

$$\phi(0)+\phi(3)+\phi(6)+\ldots+\phi(n)=\frac{E^{n+3}-1}{E^3-1}\phi(0)=\psi_3(n),$$

$$\phi(1)+\phi(4)+\phi(7)+\ldots+\phi(n)=\frac{E^{n+3}-E}{E^3-1}\phi(0)=\psi_3(n)-\lambda_1,$$

$$\phi(2)+\phi(5)+\phi(8)+\ldots+\phi(n)=\frac{E^{n+3}-E^2}{E^3-1}\phi(0)=\psi_3(n)-\lambda_2,$$

where λ_1 and λ_2 are constants,

$$\lambda_1=\frac{E-1}{E^3-1}\phi(0)=\frac{1}{E^2+E+1}\phi(0),$$

$$\lambda_2=\frac{E^2-1}{E^3-1}\phi(0)=\frac{E+1}{E^2+E+1}\phi(0).$$

Should $\phi(n)$ contain only even powers of n, we may replace E by E^{-1}, and therefore

$$\lambda_1=\frac{1}{E^2+E+1}\phi(0)=\frac{E^2}{E^2+E+1}\phi(0),$$

and $$\lambda_1+\lambda_2=\frac{E^2+E+1}{E^2+E+1}\phi(0)=\phi(0).$$

4. Generally, we have

$$\phi(0)\quad+\phi(q)\quad+\phi(2q)\quad+\ldots+\phi(n)=\frac{E^{n+q}-1}{E^q-1}\phi(0)=\psi_q(n),$$

$$\phi(1)\quad+\phi(q+1)\quad+\phi(2q+1)+\ldots+\phi(n)=\psi_q(n)-\lambda_1,$$

$$\phi(2)\quad+\phi(q+2)\quad+\phi(2q+2)+\ldots+\phi(n)=\psi_q(n)-\lambda_2,$$

....,

$$\phi(q-1)+\phi(2q-1)+\phi(3q-1)+\ldots+\phi(n)=\psi_q(n)-\lambda_{q-1},$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{q-1}$ are constants, such that $\lambda_r=\dfrac{E^r-1}{E^q-1}\phi(0)$.

Should $\phi(n)$ contain only even powers of n, we may replace E by E^{-1},

$$\lambda_r=\frac{E^r-1}{E^q-1}\phi(0)=\frac{E^{-r}-1}{E^{-q}-1}\phi(0)=\frac{E^q-E^{q-r}}{E^q-1}\phi(0)$$

$$=\phi(0)-\frac{E^{q-r}-1}{E^q-1}\phi(0)=\phi(0)-\lambda_{q-r}.$$

Hence $\phi(0)=\lambda_1+\lambda_{q-1}=\lambda_2+\lambda_{q-2}=\ldots=\lambda_r+\lambda_{q-r}=\ldots.$

But if $\phi(n)$ contain only odd powers of n, when we replace E by $\frac{1}{E}$, a change of sign is introduced; in this case $\phi(0)$ is necessarily equal to 0,

$$\lambda_r = \frac{E^r - 1}{E^q - 1}\phi(0) = -\frac{E^q - E^{q-r}}{E^q - 1}\phi(0)$$
$$= -\phi(0) + \lambda_{q-r} = \lambda_{q-r}.$$

Hence $\qquad \lambda_1 = \lambda_{q-1},\ \lambda_2 = \lambda_{q-2},\ \ldots,$ &c.

5. Whatever be the form of $\phi(n)$, provided it be rational integral and finite,

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{q-1} = \frac{E^{q-1} + E^{q-2} + \ldots - (q-1)}{E^q - 1}\phi(0)$$
$$= \left[\frac{1}{E-1} - \frac{q}{E^q - 1}\right]\phi(0)$$
$$= \left[\frac{1}{e^D - 1} - \frac{q}{e^{qD} - 1}\right]\phi(0)$$
$$= \left[\frac{q-1}{2} - \frac{q^2-1}{2!}B_1 D + \frac{q^4-1}{4!}B_2 D^3 - \frac{q^6-1}{6!}B_3 D^5 \ldots\right]\phi(0).$$

Hence if $\qquad \phi(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3 \ldots,$

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{q-1}$$
$$= \frac{q-1}{2}a_0 - \frac{q^2-1}{2}B_1 a_1 + \frac{q^4-1}{4}B_2 a_3 - \frac{q^6-1}{6}B_3 a_5 \ldots.$$

Moreover, if s be any factor of q,

$$s(\lambda_s + \lambda_{2s} + \lambda_{3s} + \ldots + \lambda_{q-s}) = \frac{q-s}{2}a_0 - \frac{q^2-s^2}{2}B_1 a_1 + \frac{q^4-s^4}{4}B_2 a_3 \ldots.$$

6. The value of the constant λ can now be found in certain cases.

It follows from (4) that when $\phi(n)$ contains only even powers of n, if

$$\phi(q) + \phi(2q) + \phi(3q) + \ldots + \phi(n) = F(n),$$

then $\phi(r) + \phi(q+r) + \phi(q+2r) + \ldots + \phi(n_1)$
$$+ \phi(q-r) + \phi(2q-r) + \ldots + \phi(n_2) = F(n_1) + F(n_2),$$

and when $\phi(n)$ contains only odd powers, if

$$\phi(r) + \phi(q+r) + \phi(2q+r) + \ldots + \phi(n) = f(n),$$

then $\phi(q-r) + \phi(2q-r) + \phi(3q-r) + \ldots + \phi(n) = f(n)$ also.

Again when $q=2$, it has already been shown that, if

$$\phi(n)=a_0+a_1n+a_2n^2+\ldots,$$

$$\lambda=\frac{a_0}{2}-\frac{2^2-1}{2}B_1a_1+\frac{2^4-1}{4}B_2a_3-\frac{2^6-1}{6}B_3a_5+\ldots.$$

Next suppose $\phi(n)$ to contain only odd powers of n, then

$$\phi(n)=b_1n+b_2n^3+b_3n^5+b_4n^7+\ldots.$$

Let $q=3$,

$$\lambda_1=\lambda_2=-\tfrac{1}{2}\left(\frac{3^2-1}{2}B_1b_1-\frac{3^4-1}{4}B_2b_2+\frac{3^6-1}{6}B_3b_3\ldots\right).$$

Let $q=4$,

$$2\lambda_2=-\left(\frac{4^2-2^2}{2}B_1b_1-\frac{4^4-2^4}{4}B_2b_2+\frac{4^6-2^6}{6}B_3b_3\ldots\right),$$

and

$$2\lambda_1+\lambda_2=-\left(\frac{4^2-1^2}{2}B_1b_1-\frac{4^4-1}{4}B_2b_2+\frac{4^6-1}{6}B_3b_3\ldots\right).$$

Therefore

$$4\lambda_1=4\lambda_3=-\left(\frac{4^2+2^2-2}{2}B_1b_1-\frac{4^4+2^4-2}{4}B_2b_2+\frac{4^6+2^6-2}{6}B_3b_3\ldots\right).$$

Let $q=6$, $$3\lambda_3=\Sigma(-1)^r\frac{6^{2r}-3^{2r}}{r}B_rb_r,$$

$$2(\lambda_2+\lambda_4)=\Sigma(-1)^r\frac{6^{2r}-2^{2r}}{r}B_rb_r,$$

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5=\Sigma(-1)^r\frac{6^{2r}-1}{r}B_rb_r;$$

therefore $$\lambda_3=\tfrac{1}{3}\Sigma(-1)^r\frac{6^{2r}-3^{2r}}{r}B_rb_r,$$

$$\lambda_2=\lambda_4=\tfrac{1}{4}\Sigma(-1)^r\frac{6^{2r}-2^{2r}}{r}B_rb_r.$$

$$\lambda_1=\lambda_5=\tfrac{1}{12}\Sigma(-1)^r\frac{6^{2r}+2.3^{2r}+3.2^{2r}-6}{r}B_rb_r,$$

and similarly for higher values of q.

As particular cases, we have, if

$$3^{2p-1}+6^{2p-1}+9^{2p-1}+\ldots+n^{2p-1}=f(n),$$

then $$1^{2p-1}+4^{2p-1}+7^{2p-1}+\ldots+n^{2p-1}=f(n)-(-1)^p\frac{3^{2p}-1}{4p}B_p,$$

and $$2^{2p-1}+5^{2p-1}+8^{2p-1}+\ldots+n^{2p-1}=f(n)-(-1)^p\frac{3^{2p}-1}{4p}B_p.$$

If $$4^{2p-1}+8^{2p-1}+12^{2p-1}+\ldots+n^{2p-1}=f(n),$$

then $$2^{2p-1}+6^{2p-1}+\ldots+n^{2p-1}=f(n)-(-1)^p\frac{4^{2p}-2^{2p}}{4p}B_p,$$

$$1^{2p-1}+5^{2p-1}+\ldots+n^{2p-1}=f(n)-(-1)^p\frac{4^{2p}+2^{2p}-2}{8p}B_p,$$

$$3^{2p-1}+7^{2p-1}+\ldots+n^{2p-1}=f(n)-(-1)^p\frac{4^{2p}+2^{2p}-2}{8p}B_p.$$

MERSENNE'S NUMBERS.

By *W. W. Rouse Ball.*

AMONG the unsolved riddles of higher arithmetic is the discovery of the method by which Mersenne determined values of p which make 2^p-1 a prime. This formed the subject of a note by M. Lucas in 1878 in the *Messenger of Mathematics*, Vol. VII, and it may be interesting if I sum up here the facts established at present, as far as I know them.

2. Mersenne asserted (*Cogitata Physico-mathematica*, Paris, 1644, Praefatio generalis, Art. 19) that the only values of p, not greater than 257, which make 2^p-1 a prime are 1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257: to which list Herr Seelhoff has shown that we must add 61.

3. This has been verified for all except twenty-three values of p; namely, 67, 71, 89, 101, 103, 107, 109, 127, 137, 139, 149, 157, 163, 167, 173, 181, 193, 197, 199, 227, 229, 241, 257. It remains to prove that $p=67$, $p=127$, and $p=257$ make 2^p-1 prime, and that the other values of p here given make 2^p-1 composite.

4. It is evident that if p is not a prime then 2^p-1 is composite, and two or more of its factors can be written down by inspection. I confine myself therefore to prime values of p. It may be noted that if $p=12n+1$ or $12n+5$, then the last digit in 2^p-1 is 1, but if $p=12n-1$ or $12n-5$, then the last digit in 2^p-1 is 7.