If $$4^{2p-1}+8^{2p-1}+12^{2p-1}+\ldots+n^{2p-1}=f(n),$$

then $$2^{2p-1}+6^{2p-1}+\ldots+n^{2p-1}=f(n)-(-1)^p\frac{4^{2p}-2^{2p}}{4p}B_p,$$

$$1^{2p-1}+5^{2p-1}+\ldots+n^{2p-1}=f(n)-(-1)^p\frac{4^{2p}+2^{2p}-2}{8p}B_p,$$

$$3^{2p-1}+7^{2p-1}+\ldots+n^{2p-1}=f(n)-(-1)^p\frac{4^{2p}+2^{2p}-2}{8p}B_p.$$

MERSENNE'S NUMBERS.

By *W. W. Rouse Ball.*

AMONG the unsolved riddles of higher arithmetic is the discovery of the method by which Mersenne determined values of p which make 2^p-1 a prime. This formed the subject of a note by M. Lucas in 1878 in the *Messenger of Mathematics*, Vol. VII, and it may be interesting if I sum up here the facts established at present, as far as I know them.

2. Mersenne asserted (*Cogitata Physico-mathematica*, Paris, 1644, Praefatio generalis, Art. 19) that the only values of p, not greater than 257, which make 2^p-1 a prime are 1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257: to which list Herr Seelhoff has shown that we must add 61.

3. This has been verified for all except twenty-three values of p; namely, 67, 71, 89, 101, 103, 107, 109, 127, 137, 139, 149, 157, 163, 167, 173, 181, 193, 197, 199, 227, 229, 241, 257. It remains to prove that $p=67$, $p=127$, and $p=257$ make 2^p-1 prime, and that the other values of p here given make 2^p-1 composite.

4. It is evident that if p is not a prime then 2^p-1 is composite, and two or more of its factors can be written down by inspection. I confine myself therefore to prime values of p. It may be noted that if $p=12n+1$ or $12n+5$, then the last digit in 2^p-1 is 1, but if $p=12n-1$ or $12n-5$, then the last digit in 2^p-1 is 7.

5. The following table gives—so far as I am aware—all the results known at present. The cases that still await verification are marked with an asterisk.*

p.	*Value of* 2^p-1.		
1	1	prime	
2	3	prime	
3	7	prime	
5	31	prime	
7	127	prime	
11	2047=23×89	composite	
13	8191	prime	
17	131071	prime	
19	524287	prime	
23	8388607=47×178481	composite	Fermat (see Art. 7).
29	536870911=233×1103×2089	composite	Plana (see Art. 10).
31	2147483647	prime	Euler (see Art. 8).
37	137438953471=223×616318177	composite	Fermat (see Art. 7).
41	2199023255551=13367×164511353	composite	Plana (see Art. 9).
43	8796093022207=431×9719×2099863	composite	Landry (see Art. 12).
47	140737488355327 =2351×4513×13264529	composite	Landry (see Art. 12).
53	9007199254740991 =6361×69431×20394401	composite	Landry (see Art. 12).
59	576460752303423487 =179951×3203431780337	composite	Landry (see Art. 12).
61	2305843009213693951	prime	Seelhoff (see Art. 13).
67	139573952589676412927	*	
71	2233183241434822606847	*	
73	≡0 (439)	composite	Le Lasseur (see Art. 14).
79	≡0 (2687)	composite	Le Lasseur (see Art. 14).
83	≡0 (167)	composite	Lucas (see Art. 11).
89	618970019517690187449562111	*	
97	≡0 (11447)	composite	Le Lasseur (see Art. 16).
101	2535301200456459007793406410751	*	
103	....	*	
107	....	*	
109	....	*	
113	≡0 (3391)	composite	Le Lasseur (see Art. 14).
127	170141183332488431731047303715884361727	*	
131	≡0 (263)	composite	Lucas (see Art. 11).
137	....	*	
139	....	*	
149	....	*	
151	≡0 (18121)	composite	Le Lasseur (see Art. 16).
157	....	*	
163	....	*	
167	....	*	
173	....	*	
179	≡0 (359)	composite	Lucas (see Art. 11).
181	....	*	
191	≡0 (383)	composite	Lucas (see Art. 11).
193	....	*	
197	....	*	
199	....	*	
211	≡0 (15193)	composite	Le Lasseur (see Art. 16).
223	≡0 (18287)	composite	Le Lasseur (see Art. 16).
227	....	*	
229	....	*	
233	≡0 (1399)	composite	Le Lasseur (see Art. 15).
239	≡0 (479)	composite	Lucas (see Art. 11).
241	....	*	
251	≡0 (503)	composite	Lucas (see Art. 11).
257	....	*	

6. I proceed to explain how some of these results were established. The factors (if any) of such values of $2^p - 1$ as are less than a million can be verified easily: they have been known for a long time, and I need not allude to them in detail.

7. The factors of $2^p - 1$ when $p = 11$, 23, 37 had been indicated by Fermat, some four years prior to the publication of Mersenne's work, in a letter, dated Oct. 18, 1640 (*Opera Mathematica*, Toulouse, 1679, p. 164; or Brassinne's *Précis*, Paris, 1853, p. 144). The passage is as follows: "En la progression double, si d'un nombre quarré, generalement parlant, vous ôtez 2 ou 8 ou 32 &c., les nombres premiers, moindres de l'unité qu'un multiple du quaternaire qui mesureront le reste, feront l'effet requis. Comme de 25, qui est un quarré, ôtez 2, le reste 23 mesurera la 11 puissance – 1; ôtez 2 de 49, le reste 47 mesurera la 23 puissance – 1. Otez 2 de 225, le reste 223 mesurera la 37 puissance – 1, &c."

8. The fact that $2^p - 1$ is prime when $p = 31$ was established by Euler in 1771 (*Historie de l'Académie des Sciences* for 1772, Berlin, 1774, p. 36). Fermat had asserted in the letter mentioned in the last article that the only possible prime factors of $2^p \pm 1$, when p was prime, were of the form $np + 1$, where n is an integer or zero. This was proved by Euler in 1748 (*Commentationes Arithmeticae Collectae*, St. Petersburg, 1849, Vol. I. pp. 55, 56) who added that since $2^p \pm 1$ is odd, every factor of it must be odd, and therefore if p is odd n must be even. But if p is a given number we can define n much more closely, and Euler showed that the prime factors (if any) of $2^{31} - 1$ were necessarily primes of the form $248n + 1$ or $248n + 63$; also they must be less than $\sqrt{(2^{31} - 1)}$, that is, than 46339. Hence it is necessary to try only forty divisors to see if $2^{31} - 1$ is a prime or composite.

Euler's note was suggested by John Bernoulli's memoir on the factors of numbers in the denary scale of the form $10^p \pm 1$ (*Mémoires de l'Academie des Sciences* for 1771, Berlin, 1773, p. 318). Mersenne's statement refers to numbers of a similar form in the binary scale.

9. The factors of $2^p - 1$ when $p = 41$ were given by Plana in 1859 (*Memorie della Reale Accademia delle Scienze di Torino*, Series 2, Vol. XX, 1863, p. 130). The prime factors (if any) are primes of the form $328n + 1$ or $328n + 247$, and lie between 1231 and $\sqrt{(2^{41} - 1)}$, that is, 1048573. Hence it is

necessary to try only 513 divisors to see if $2^{41}-1$ is composite: the seventeenth of these divisors gives the required factors.

Plana adds (p. 137) the forms of the prime divisors of 2^p-1, if p is not greater than 101. This is Euler's method of attacking the problem, but it is too laborious to use for values of p greater than 41.

10. The factors of 2^p-1 when $p=29$ are quoted by Plana in the memoir last cited (p. 138). I believe they were discovered by Euler, but I cannot lay my hands on any reference earlier than this memoir.

11. The fact that 2^p-1 is composite for the values $p=83$, 131, 179, 191, 239, 251 follows from a proposition due to M. Lucas (*American Journal of Mathematics*, 1878, Vol. I., p. 236) to the effect that if $4n+3$ and $8n+7$ are primes then $2^{4n+3}-1\equiv 0$ (mod. $8n+7$). The proposition covers the cases of $p=11$ and $p=23$, and a demonstration of it for these values of p was given by Plana in his memoir (p. 138), though he does not seem to have noticed that the method of proof was general.

The proof of Lucas's theorem is as follows. We have, by Fermat's theorem,

$$2^{8n+6}-1\equiv 0 \;\; (8n+7),$$

therefore $$(2^{4n+3}-1)(2^{4n+3}+1)\equiv 0 \;\; (8n+7).$$

But Fermat showed that, if $4n+3$ is prime, then the second factor of the left-hand side is not divisible by $8n+7$, therefore

$$2^{4n+3}-1\equiv 0 \;\; (8n+7).$$

12. The discovery of the factors of 2^p-1 for the values $p=43$, 47, 53, 59 is due apparently to the late M. F. Landry, who established theorems on the factors, if any, of numbers of certain forms. Instead of publishing his results he issued a challenge to all mathematicians to solve the general problem. This is contained in a pamphlet, Paris, 1867, in which the factors of certain numbers are given, and he implies (p. 8) that he had obtained the factors of 2^p-1 when $p=43$, 47, 53, 59. He seems to have communicated his results to M. Lucas, who quotes them in the above-mentioned memoir on p. 236.

13. The fact that 2^p-1 is prime when $p=61$ was conjectured by M. Landry (Lucas, p. 238) and has been demonstrated by Herr Seelhoff (*Zeitschrift für Mathematik und Physik*, 1886, Vol. XXI., p. 178). The value of $2^{61}-1$ contains 19 digits, and at present is the highest number known to be a prime. According to Mersenne $2^{67}-1$, $2^{127}-1$, and $2^{257}-1$ are also primes: they contain respectively, 21, 39, and 78 digits.

14. The factors of 2^p-1 when $p=73, 79, 113$ were given first by M. Le Lasseur, and are quoted by M. Lucas in the above cited memoir (p. 236).

15. A factor of 2^p-1 when $p=233$ was discovered later by M. Le Lasseur, and is quoted by M. Lucas (*Récréations*, 1882, Vol. I., p. 241).

16. The factors of 2^p-1 when $p=97, 151, 211, 223$ were determined subsequently by M. Le Lasseur (Lucas, *Récréations*, 1883, Vol. II., p. 230).

I do not know how M. Le Lasseur proved these results or where he published them. I understand that they have been verified.

17. Mersenne's results are connected closely with the theory of *perfect numbers.*

18. It has been known since the time of Euclid (*Elements*, Book IX., Prop. 36) that any number of the form $2^{p-1}(2^p-1)$, where 2^p-1 is a prime, is perfect, that is, is such that it is equal to the sum of its integral sub-divisors.

It is probable that an odd number cannot be perfect, though a rigorous demonstration of this has not been given hitherto. Euclid's formula includes all even perfect numbers (Euler, *Commentationes Arithmeticae Collectae*, St Petersburg, 1849, Vol. II., p. 514, Art. 107: Professor Sylvester has published an analysis of the argument in *Nature*, Dec. 15, 1887, Vol. XXXVII., p. 152.) Hence it is believed that Euclid's formula includes all perfect numbers.

19. If 2^p-1 is a prime, it follows that p is a prime; hence either the last digit of an even perfect number must be 6, or the last two digits must be 28; also every even perfect number (except 6) is congruent to unity to the modulus 9.

20. Using the results of Art. 2, we see that the first nine even perfect numbers are obtained by putting $p=2$, 3, 5, 7, 13, 17, 19, 31, 61, in $2^{p-1}(2^p-1)$; and are 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2658455989570131744644692615953846176.

21. After making the assertion given above in Art. 2, Mersenne continued as follows. "Qui vndecim alios repererit, nouerit se analysim omnem, quae fuerit hactenus, superasse: memineritque interea nullum esse perfectum à 17000 potestate ad 32000; & nullum potestatum interuallum tantum assignari posse, quin detur illud absque perfectis. Verbi gratia, si fuerit exponens 1050000, nullus erit numerus progressionis duplae vsque ad 2090000, qui perfectis numeris seruiat, hoc est qui minor vnitate, primus existat. Vnde clarum est quàm rari sint perfecti numeri, & quàm meritò viris perfectis comparentur; esseque vnam ex maximis totius Matheseos difficultatibus, praescriptam numerorum perfectorum multitudinum exhibere; quemadmodum & agnoscere num dati numeri 15, aut 20 caracteribus constantes, sint primi necne, cùm nequidem saeculum integrum huic examini, quocumque modo hactenus cognito, sufficiat." From the last clause it would appear that he did not know how the result was demonstrated.

It may be added also that the result was not known generally to Mersenne's contemporaries, for even so well read a mathematician as Oughtred gives an incorrect rule (*Mathematicall Recreations*, London, 1653, p. 92).

I am inclined to believe that the statement is due originally to Fermat, in which case the communication to Mersenne and the absence of a demonstration would be what we should expect. Moreover Fermat had investigated rules for determining whether a number was prime, and I think it probable that he had discovered certain theorems on the subject which now are lost. Thus in a letter, dated April 7, 1643 (*i.e.* a year before Mersenne published the above assertions) Fermat writes to Mersenne: "Vous me demandez si le nombre 100895598169 est premier ou non, et une méthode pour decouvrir dans l'espace d'un jour s'il est premier ou composé. A cette question, je réponds que ce nombre est composé et se fait du produit de ces deux: 898423 et 112303, qui sont premiers." (Lucas, p. 230).

As far as I know the only remark of Fermat which bears directly on the problem of Mersenne's numbers is the one referred to above in Art. 8: namely, that the prime factors of $2^p - 1$ are of the form $np+1$. I think it probable that Fermat was aware of the forms of n corresponding to various numerical values of p (see Art. 9); and it is possible that these are particular or partial cases of some general theorem, from which the results enunciated by Mersenne are deducible. It is manifest, however, that Fermat's remark would be insufficient by itself to enable anyone either to answer the question asked by Mersenne or to prove the assertion given above in Art. 2; nor are such propositions on the determination of whether a given number is prime as have been enunciated subsequently sufficient for these purposes. Hence the riddle as to how Mersenne's numbers were discovered remains unsolved.

NOTE ON A FORMULA IN SPHERICAL HARMONICS.

By *R. Fujisawa*.

It is well-known that

$$\frac{1-\alpha^2}{1-2\alpha\cos\gamma+\alpha^2} = 1 + 2\alpha\cos\gamma + 2\alpha^2\cos 2\gamma + \ldots + 2\alpha^n\cos n\gamma + \ldots.$$

If $(1-2\alpha\cos\gamma+\alpha^2)^{-\frac{1}{2}}$ be expanded in a series of ascending powers of α, the coefficient of α^n is denoted by $P_n(\cos\gamma)$. Thus, by definition, we have

$$\frac{1}{\sqrt{(1-2\alpha\cos\gamma+\alpha^2)}} = P_0 + P_1\alpha + P_2\alpha^2 + \ldots + P_n\alpha^n + \ldots.$$

Squaring,

$$\frac{1}{1-2\alpha\cos\gamma+\alpha^2} = Q + Q_1\alpha + Q_2\alpha^2 + \ldots + Q_n\alpha^n + \ldots,$$

where $$Q_n = \begin{array}{ll} 2P_0P_n + 2P_1P_{n-1} + \ldots + 2P_{\frac{1}{2}(n-1)}P_{\frac{1}{2}(n+1)} & (n \text{ odd}), \\ 2P_0P_n + 2P_1P_{n-1} + \ldots + 2P^2_{\frac{1}{2}n} & (n \text{ even}). \end{array}$$