satisfies the four conditions $s_1 \equiv 0 \equiv s_2 \equiv s_3 \equiv s_5$, and the conditions $s_1 \equiv 0 \equiv s_3 \equiv s_4 \equiv s_5$ hold good for the quantic whose roots are the cubes of the last. Thus we see that as far as the simplest conditions are concerned, four of the conditions $s_1 \equiv 0 \equiv s_2 \equiv s_3 \equiv s_4 \equiv s_5$ are not sufficient to make all the roots different.

On the other hand, we may choose five of Hermite's conditions which shall not necessitate the other four, as in the case where the congruence only contains odd powers of z, so that $s_1 \equiv 0 \equiv s_3 \equiv s_5 \equiv s_7 \equiv s_9$. Since, however, it is evident that the lowest powers give the simplest relations, it seems scarcely worth while to investigate whether any four more complicated ones such as $s_1 \equiv 0 \equiv s_3 \equiv s_7 \equiv s_8$ are sufficient to include all the other conditions.

EXPRESSION FOR THE SUM OF THE CUBES OF THE DIVISORS OF A NUMBER IN TERMS OF PARTITIONS OF INFERIOR NUMBERS.

By *J. W. L. Glaisher.*

It was shown by Euler that, if $P(n)$ denote the number of partitions of n into the numbers 1, 2, 3, ..., repetitions not excluded, and if $P(0)$ have the value unity, then

$$P(n) - P(n-1) - P(n-2) + P(n-5) + P(n-7) - \&c. = 0,$$

where 1, 2, 5, 7, ..., are the pentagonal numbers $\frac{1}{2}(3r^2 \pm r)$ and the signs of the term are positive or negative according as r is even or uneven.*

It is easy to show that

$$P(n-1) + 2P(n-2) - 5P(n-5) - 7P(n-7) + \&c. = \sigma(n),$$

where $\sigma(n)$ denotes the sum of the divisors of n.

I have also found that

$$P(n-1) + 2^3P(n-2) - 5^3P(n-5) - 7^3P(n-7) + \&c.$$
$$= -\tfrac{1}{12}\{5\sigma_3(n) - (18n-1)\,\sigma(n)\},$$

where $\sigma_3(n)$ denotes the sum of the cubes of the divisors of n.

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* Euler, *Commentationes Arithmeticæ Collectæ*, Vol. i., p. 91. See also *Proc. Lond. Math. Soc.*, Vol. xxi., p. 202.

The pentagonal numbers (which are the thirds of these triangular numbers which are divisible by 3) are 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77, ..., and in the last two formulæ the term $P\{n-\frac{1}{2}(3r^2 \pm r)\}$ has the sign $(-1)^{r-1}$.

Thus, p denoting any pentagonal number and the summation extending from $p=1$ to the $p=$ the pentagonal number next inferior to n, we have

$$\Sigma \pm P(n-p) = P(n),$$

$$\Sigma \pm pP(n-p) = \sigma(n),$$

$$\Sigma \pm p^2 P(n-p) = -\tfrac{1}{12}\{5\sigma_3(n) - (18n-1)\sigma(n)\}.$$

In all these formulæ $P(0)$ is to have the value unity.

As an example, putting $n=15$, the formulæ give

$$P(14) + P(13) - P(10) - P(8) + P(3) + P(0) = P(15),$$

$$P(14) + 2P(13) - 5P(10) - 7P(8) + 12P(3) + 15P(0) = \sigma(15),$$

$$P(14) + 2^2P(13) - 5^2P(10) - 7^2P(8) + 12^2P(3) + 15^2P(0)$$

$$= -\tfrac{1}{12}\{5\sigma_3(15) - 269\sigma(15)\},$$

which are easily verified since $P(14)=135$, $P(13)=101$, $P(10)=42$, $P(8)=22$, $P(3)=3$, and $P(0)=1$, the three formulæ giving respectively the values 176, 24 and -932.

Using the above notation we may express $\sigma_3(n)$ in terms of partitions by the formula

$$\sigma_3(n) = \tfrac{1}{5}\Sigma \pm \{p(18n-p-1)P(n-p)\}.^*$$

Thus in the above example

$$5\sigma_3(15) = \Sigma \pm p(269-p)P(n-p)$$

$$= 257P(14) + 490P(13) - 1045P(10) - 1295P(8)$$

$$+ 1500P(3) + 1335P(0).$$

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* The values of $P(n)$ up to $n=59$ are given by Euler, *loc. cit.* On account of the fundamental character of the function $P(n)$, it seems to be always a matter of interest to express other arithmetical functions in terms of P's.
