EXPRESSIONS FOR SYMMETRICAL FUNCTIONS OF I, G, E IN TERMS OF q.

By *J. W. L. Glaisher.*

In connexion with the preceding paper it seems desirable to state the values of the symmetrical functions of I, G, E which are referred to in § 24 (p. 62).

Notation, § 1.

§ 1. The quantities I and G denote $E-K$ and $E-k'^2K$ respectively* and

$$i=\frac{2KI}{\pi^2},\quad g=\frac{2KG}{\pi^2},\quad e=\frac{2KE}{\pi^2}.$$

Square brackets are used to denote the sum of all the terms of the type of the one included by them: thus

$[i]$ stands for $i+g+e$,

$[i^2g]$ „ „ $i^2g+g^2e+e^2i+i^2e+e^2g+g^2i$,

$[i^2ge]$ „ „ $i^2ge+g^2ei+e^2ig$.

In all cases the symbol of summation Σ refers to the letter n which is to have all values from 1 to ∞.

Values of $[i]$, $[ig]$, *&c. in terms of* q, § 2.

§ 2. *First order.*

$$[i]=1-24\Sigma\sigma(n)\,q^{2n}.$$

Second order.

$$[i^2]=1+240\Sigma\sigma_3(n)\,q^{2n}-96\Sigma n\sigma(n)\,q^{2n},$$

$$[ig]=-96\Sigma n\sigma(n)\,q^{2n}.$$

* See *Quart. Math. Jour.*, Vol. xx., p. 352, and *Proc. Camb. Phil. Soc.*, Vol. v., p. 191.

Third order.

$$[i^3] = 1 - 504\Sigma\sigma_5(n)\,q^{2n} + 720\Sigma n\sigma_3(n)\,q^{2n} - 192\Sigma n^2\sigma(n)\,q^{2n},$$

$$[i^2g] = \quad 480\Sigma n\sigma_3(n)\,q^{2n} - 384\Sigma n^2\sigma(n)\,q^{2n},$$

$$ige = \quad -64\Sigma n^2\sigma(n)\,q^{2n}.$$

Fourth order.

$$[i^4] = 1 + 480\Sigma\sigma_7(n)\,q^{2n} - 1344\Sigma n\sigma_5(n)\,q^{2n} + 1152\Sigma n^2\sigma_3(n)\,q^{2n} - 256\Sigma n^3\sigma(n)\,q^{2n},$$

$$[i^3g] = -672\Sigma n\,\sigma_5(n)\,q^{2n} + 1152\Sigma n^2\sigma_3(n)\,q^{2n} - 512\Sigma n^3\sigma(n)\,q^{2n},$$

$$[i^2g^2] = \quad 384\Sigma n^2\sigma_3(n)\,q^{2n} - 256\Sigma n^3\sigma(n)\,q^{2n},$$

$$[i^2ge] = \quad 192\Sigma n^2\sigma_3(n)\,q^{2n} - 256\Sigma n^3\sigma(n)\,q^{2n}.$$

Fifth order.

$$[i^5] = 1 - 264\Sigma\sigma_9(n)\,q^{2n} + 1200\Sigma n\sigma_7(n)\,q^{2n} - 1920\Sigma n^2\sigma_5(n)\,q^{2n} + 1280\Sigma n^3\sigma_3(n)q^{2n} - 256\Sigma n^5\sigma(n)q^{2n},$$

$$[i^4g] = \quad 480\Sigma n\sigma_7(n)\,q^{2n} - 1536\Sigma n^2\sigma_5(n)\,q^{2n} + 1536\Sigma n^3\sigma_3(n)\,q^{2n} - 512\Sigma n^4\sigma(n)\,q^{2n},$$

$$[i^3g^2] = -384\Sigma n^2\sigma_5(n)\,q^{2n} + 1024\Sigma n^3\sigma_3(n)\,q^{2n} - 512\Sigma n^4\sigma(n)\,q^{2n},$$

$$[i^3ge] = -192\Sigma n^2\sigma_5(n)\,q^{2n} + 384\Sigma n^3\sigma_3(n)\,q^{2n} - 256\Sigma n^4\sigma(n)\,q^{2n},$$

$$[i^2g^2e] = \quad 256\Sigma n^3\sigma_3(n)\,q^{2n} - 256\Sigma n^4\sigma(n)\,q^{2n}.$$

§ 3. From these formulæ, we find

I.

$$24\Sigma\ \sigma(n)\,q^{2n} = 1 - [i],$$

$$96\Sigma n\,\sigma(n)\,q^{2n} = \ -[ig],$$

$$64\Sigma n^2\sigma(n)\,q^{2n} = -ige,$$

$$256\Sigma n^3\sigma(n)\,q^{2n} = [i^2g^2] - 2[i^2ge],$$

$$256\Sigma n^4\sigma(n)\,q^{2n} = [i^3g^2] - 2[i^2ge] - [i^2g^2e].$$

II.

$$240\Sigma\ \sigma_3(n)q^{2n} = -1 + [i^2] - [ig],$$
$$480\Sigma n\,\sigma_3(n)q^{2n} = [i^2g] - 6ige,$$
$$192\Sigma n^2\sigma_3(n)q^{2n} = [i^2g^2] - [i^2ge],$$
$$256\Sigma n^3\sigma_3(n)q^{2n} = [i^3g^2] - 2[i^3ge].$$

III.

$$1008\Sigma\ \sigma_5(n)q^{2n} = 2 - 2[i^3] + 3[i^2g] - 12ige,$$
$$672\Sigma n\,\sigma_5(n)q^{2n} = -[i^3g] - 2[i^2ge] + 4[i^2g^2],$$
$$384\Sigma n^2\sigma_5(n)q^{2n} = [i^3g^2] - 4[i^3ge] + 2[i^2g^2e].$$

IV.

$$480\Sigma\ \sigma_7(n)q^{2n} = -1 + [i^4] + 3[i^2g^2] - 2[i^3g],$$
$$480\Sigma n\,\sigma_7(n)q^{2n} = [i^4g] - 8[i^3ge] + 6[i^2g^2e].$$

V.

$$528\Sigma\ \sigma_9(n)q^{2n} = 2 - 2[i^5] + 5[i^4g] - 2[i^3g^2] - 16[i^3ge] + 12[i^2g^2e].$$

Values of the first five powers of $[i]$, § 4.

§ 4. Since

$$[i]^2 = [i^2] + 2[ig],$$
$$[i]^3 = [i^3] + 3[i^2g] + 6ige,$$
&c., &c.,

we deduce from the formulæ in § 2 the following values of the first five powers of $i+g+e$,

$$[i] = 1 - 24\ \Sigma\sigma\ (n)q^{2n},$$
$$[i]^2 = 1 + 240\Sigma\sigma_3(n)q^{2n} - 288\ \Sigma n\sigma\ (n)q^{2n},$$
$$[i]^3 = 1 - 504\Sigma\sigma_5(n)q^{2n} + 2160\Sigma n\sigma_3(n)q^{2n} - 1728\Sigma n^2\sigma(n)q^{2n},$$
$$[i]^4 = 1 + 480\Sigma\sigma_7(n)q^{2n} - 4032\Sigma n\sigma_5(n)q^{2n} + 10368\Sigma n^2\sigma_3(n)q^{2n} - 6912\Sigma n^3\sigma(n)q^{2n},$$
$$[i]^5 = 1 - 264\Sigma\sigma_9(n)q^{2n} + 3600\Sigma n\sigma_7(n)q^{2n} - 17280\Sigma n^2\sigma_5(n)q^{2n} + 34560\Sigma n^3\sigma_3(n), - 20736\Sigma n^4\sigma(n)q^{2n}.$$

We may also write these formulæ

$$[i] = 1 - 24\Sigma\sigma(n)\,q^{2n},$$
$$[i]^2 = 1 + 48\Sigma\{5\sigma_3(n) - 6\,n\sigma(n)\}\,q^{2n},$$
$$[i]^3 = 1 - 72\Sigma\{7\sigma_5(n) - 30\,n\sigma_3(n) + 24n^2\sigma(n)\}\,q^{2n},$$
$$[i]^4 = 1 + 96\Sigma\{5\sigma_7(n) - 42\,n\sigma_5(n) + 108n^2\sigma_3(n) - 72n^3\sigma(n)\}\,q^{2n},$$
$$[i]^5 = 1 - 24\Sigma\{11\sigma_5(n) - 150n\sigma_7(n) + 720n^2\sigma_5(n) - 1440n^3\sigma_3(n) + 864n^4\sigma(n)\}\,q^{2n}.$$

§ 5. The formulæ contained in the preceding section were given in the *Messenger* for July 1885, and were there used to obtain the square, cube, fourth and fifth powers of the series,

$$\sigma(1)\,x + \sigma(2)\,x^2 + \sigma(3)\,x^3 + \sigma(4)\,x^4 + \&c.^*$$

Some of the other formulæ contained in that paper may also be proved by means of the formulæ in §§ 2 and 3, *e.g.*,

$$12\Sigma_{r=1}^{r=n} rr'\sigma(r)\,\sigma(r') = n^2\sigma_3(n) - n^3\sigma(n),$$

(where $r' = n - r$), which occurs on p. 35.

For, by § 2,

$$\{96\Sigma n\sigma(n)\,q^{2n}\}^2 = [ig]^2$$
$$= [i^2g^2] + 2\,[i^2ge]$$
$$= 768\Sigma n^2\sigma_3(n)\,q^{2n} - 768\Sigma n^3\,\sigma(n)\,q^{2n},$$

i.e.

$$12\,\{\Sigma n\sigma(n)\,q^{2n}\}^2 = \Sigma n^2\sigma_3(n)\,q^{2n} - \Sigma n^3\sigma(n)\,q^{2n},$$

and, by equating the coefficients of q^{2n}, we obtain the above relation.

Values of q-quotients, § 6.

§ 6. I may add that the values of the q-quotients referred to on p. 62 are:

I.

$$\frac{q^{\frac{1}{4}} - 3^3q^{\frac{9}{4}} + 5^3q^{\frac{25}{4}} - 7^3q^{\frac{49}{4}} + \&c.}{q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - 7q^{\frac{49}{4}} + \&c.} = [i].$$

II.

$$\frac{q^{\frac{1}{4}} - 3^5q^{\frac{9}{4}} + 5^5q^{\frac{25}{4}} - 7^5q^{\frac{49}{4}} + \&c.}{q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - 7q^{\frac{49}{4}} + \&c.} = [i^2] + 4\,[ig].$$

* "Expressions for the first five powers of the series in which the coefficients are the sums of the divisors of the exponents," Vol. xv, p. 33.

III.

$$\frac{q^{\frac{1}{4}} - 3^7 q^{\frac{9}{4}} + 5^7 q^{\frac{25}{4}} - 7^7 q^{\frac{49}{4}} + \&c.}{q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - 7q^{\frac{49}{4}} + \&c.} = [i^3] + 9\,[i^2 g] + 48ige.$$

IV.

$$\frac{q^{\frac{1}{4}} - 3^9 q^{\frac{9}{4}} + 5^9 q^{\frac{25}{4}} - 7^9 q^{\frac{49}{4}} + \&c.}{q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - 7q^{\frac{49}{4}} + \&c.}$$
$$= [i^4] + 16\,[i^3 g] - 6\,[i^2 g^2] + 288ige.$$

V.

$$\frac{q^{\frac{1}{4}} - 3^{11} q^{\frac{9}{4}} + 5^{11} q^{\frac{25}{4}} - 7^{11} q^{\frac{49}{4}} + \&c.}{q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - 7q^{\frac{49}{4}} + \&c.}$$
$$= [i^5] + 25\,[i^4 g] - 494\,[i^3 g^2] + 1768\,[i^3 ge] + 2634\,[i^2 g^2 e].$$

ON CENTRES OF PRESSURE.

By Prof. *A. Anderson, M.A.*

The following applications of some elementary methods of finding centres of pressure may be of interest, as they are not to be met with in the ordinary text-books on Hydrostatics.

If a plane area be entirely immersed in a homogeneous liquid, and if perpendiculars be drawn to the surface from every point of its boundary, a vertical line through the centre of gravity of the volume thus enclosed will meet the plane area at its centre of pressure. This well-known theorem may be directly applied to the case of a triangle completely immersed in any way.

Let the vertices of the triangle ABC be sunk to depths h_1, h_2 and h_3; and suppose $h_1 < h_2 < h_3$. At A, B, C erect perpendiculars meeting the surface of the liquid in A', B', C', and through A draw a plane AMN parallel to $A'B'C'$, cutting the lines BB', CC' in MN; and join BN. The masses of the portions of liquid $AMNC'B'A'$, $ABNM$, $ABCN$ are plainly proportional to $3h_1$, $h_2 - h_1$, $h_3 - h_1$; and, since we are only concerned with the vertical through the centre of gravity, we may substitute, for the first, three masses each numerically equal to h_1 at the points A, B, C. Also, since the centre of gravity of a tetrahedron is coincident with that of four equal particles