in (2). Doing this, and substituting $2bc\cos A$ for $-a^2+b^2+c^2$, $2R\cos A$ for a, and so on, we get

$$\left(\frac{\cot B}{\cot C}\right)^{\frac{\cos^r A}{\sin^{r+s} A}} \left(\frac{\cot C}{\cot A}\right)^{\frac{\cos^r B}{\sin^{r+s} B}} \left(\frac{\cot A}{\cot B}\right)^{\frac{\cos^r C}{\sin^{r+s} C}} < 1,$$

where s, r are of opposite signs, and A, B, C the angles of a triangle are such that $A > B > C$, or $B > C > A$, or $C > A > B$. This method gives a large number of results of this class.

NOTE ON THE NUMERATOR OF A HARMONICAL PROGRESSION.

By *G. Osborn.*

If p is a prime number greater than 3, the numerator of the harmonical progression

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p-1}$$

is divisible by p^2, and not otherwise.

If each factor is omitted in turn from $(p-1)!$ the resulting numbers all give different remainders on division by p (as is easily seen by supposing two of the remainders alike), therefore the remainders are

$$1,\ 2,\ 3,\ \ldots,\ (p-1),$$

in some order.

If we square the original numbers, the remainders become those of the series

$$1^2,\ 2^2,\ \ldots,\ (p-1)^2.$$

But $$1^2+2^2+\ldots+(p-1)^2 \equiv 0 \pmod{p},$$

therefore $$\{(p-1)!\}^2\left\{\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{(p-1)^2}\right\} \equiv 0 \pmod{p},$$

therefore the numerator of

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(p-1)^2} \equiv 0 \pmod{p}.$$

But the numerator of

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p-1} \equiv 0 \pmod{p}$$

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(by taking the first and last in pairs, &c.), therefore the numerator of

$$\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{p-1}\right)^2-\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{(p-1)^2}\right)\equiv 0 \pmod{p},$$

or of
$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{2.4}+\ldots,$$

that is, $\pi_{p-3}\equiv 0 \pmod{p}$, where π_{p-3} means the sum of the products of the first $(p-1)$ integers taken $(p-3)$ together.

Again, we have, identically

$$(p-1)(p-2)(p-3)\ldots\{p-(p-1)\}=(p-1)!,$$

or
$$p^{p-1}-\pi_1 p^{p-2}+\ldots+\pi_{p-3}p^2-\pi_{p-2}p=0,$$

but π_{p-3} is divisible by p, therefore π_{p-3} is divisible by p^2, which proves the theorem.

It seems very unlikely that this has not been given before, but I have not been able to find it; Mr. A. C. Dixon, whom I consulted among others on this point, has obtained the result differently by employing allied numbers.

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NUMERICAL FACTORS: A THEOREM.

By *Rev. J. G. Birch, M.A.*

1. Every partition of any given number N into the sum of two others less than it can be used to throw it into the form of a continuant. Let x be any number less than N, if the fraction $\frac{x}{N-x}$ be expressed as a continued fraction thus:—

$$\frac{x}{N-x}=\frac{1}{a_0-1}+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_\omega},$$

then

$$N=\begin{vmatrix} a_0, & 1, & \vdots & \vdots & \vdots & \vdots \\ -1, & a_1, & 1, & \vdots & \vdots & \vdots \\ \vdots & -1, & a_2, & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & a_{\omega-1}, & 1 \\ \vdots & \vdots & \vdots & \vdots & -1, & a_\omega \end{vmatrix},$$

or, as is usually written for shortness,

$$N=(a_0, a_1, a_2, \ldots, a_{\omega-1}, a_\omega).$$

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