therefore $\angle\, SKM = HKO$; that is, OKM touches the confocal ellipse which passes through K. Similarly OLN touches the confocal ellipse which passes through L. These tangents intersect upon TO, which is normal to the given ellipse at T: and hence, by a well-known theorem, K, L lie on *the same* confocal.

It is now easy to infer the truth of Steiner's proposition, for if any two consecutive vertices, such as K, L, of the circumscribed polygon are not on the same confocal ellipse, we can alter the position of KTL so as to bring K, L upon the same confocal, and thus diminish the perimeter of the polygon. It is evident that at least one polygon of minimum perimeter must exist; and it follows from Poncelet's theory that there is an infinite number of such polygons: that they all have the same perimeter may be proved by considerations similar to those used in the proof of Graves's theorem.

The determination of the confocal ellipse on which the vertices lie may be effected by the aid of elliptic functions, and in the same way an expression may be found for the perimeter of any one of the series of circumscribed polygons.

Relatively to the outer ellipse, the polygon is of course an inscribed polygon of maximum perimeter. The result is rather curious from a statical point of view: namely, if an endless elastic string pass through a given number of small smooth rings free to move on a fixed smooth elliptic wire, there is an infinite number of positions of unstable equilibrium.

It may be observed that if a polygon of minimum perimeter be circumscribed to an ellipse, the points of contact of the sides will be the vertices of an inscribed polygon of maximum perimeter.

ON TWO CUBIC EQUATIONS.

By Professor *Cayley*.

STARTING from the equations

$$2 + a = b^2,$$
$$2 + b = c^2,$$
$$2 + c = a^2,$$

then eliminating b, c, we find

$$(a^4 - 4a^2 + 2)^2 - (a + 2) = 0,$$

that is
$$a^8 - 8a^6 + 20a^4 - 16a^2 - a + 2 = 0;$$

we satisfy the equations by $a=b=c$, and thence by

$$a^2-a-2=(a-2)(a+1)=0;$$

there remains a sextic equation breaking up into two cubic equations; the octic equation may in fact be written

$$(a-2)(a+1)(a^3+a^2-2a-1)(a^3-3a+1)=0,$$

and we have thus the two cubic equations

$$x^3+x^2-2x-1=0,\ x^3-3x+1=0,$$

for each of which the roots (a, b, c) taken in a proper order are such that $2+a=b^2$, $2+b=c^2$, $2+c=a^2$.

It may be remarked that starting from $y^3+y^2-2y-1=0$, $y^2=x+2$, the first equation gives $(y^3-2y)^2-(y^2-1)^2=0$, that is $y^6-5y^4+6y^2-1=0$, whence

$$(x+2)^3-5(x+2)^2+6(x+2)-1=0,$$

that is $x^3+x^2-2x-1=0$.

And similarly, starting from $y^3-3y+1=0$, $y^2=x+2$, the first equation gives $(y^3-3y)^2-1=0$, that is $y^6-6y^4+9y^2-1=0$, whence $(x+2)^3-6(x+2)^2+9(x+2)-1=0$, that is

$$x^3-3x+1=0.$$

To find the roots of the equation $x^3+x^2-2x-1=0$, taking ω an imaginary cube root of unity, and writing $\alpha=\sqrt[3]{\{7(2+3\omega)\}}$, $\beta=\sqrt[3]{\{7(2+3\omega^2)\}}$, where observe that $2+3\omega$, $2+3\omega^2$ are imaginary factors of 7, viz.

$$7=(2+3\omega)(2+3\omega^2),$$

and therefore also $\alpha^3+\beta^3=7$, $\alpha\beta=7$, then the roots of the equation are

$$3a=-1+\alpha+\beta,$$

$$3b=-1+\omega\alpha+\omega^2\beta,$$

$$3c=-1+\omega^2\alpha+\omega\beta.$$

I verify herewith the equation $a^2=2+c$, viz. this gives

$$(-1+\alpha+\beta)^2=18+3(-1+\omega^2\alpha+\omega\beta),$$

or writing herein $2\alpha\beta=14$ this is

$$\alpha^2-(2+3\omega^2)\alpha+\beta^2-(2+3\omega)\beta=0,$$

that is $\alpha^2-\frac{1}{7}\beta^3\alpha \quad +\beta^2-\frac{1}{7}\alpha^3\beta \quad =0,$

or finally $(\alpha^2+\beta^2)(1-\frac{1}{7}\alpha\beta)=0,$

satisfied in virtue of $\alpha\beta=7$.

For the second equation $x^3 - 3x + 1 = 0$, ω denoting as before, the roots are

$$a = \omega^{\frac{1}{3}} + \omega^{\frac{8}{3}}, \text{ whence } a^2 = \omega^{\frac{2}{3}} + \omega^{\frac{7}{3}} + 2, = 2 + c,$$

$$b = \omega^{\frac{4}{3}} + \omega^{\frac{5}{3}}, \quad ,, \quad b^2 = \omega^{\frac{8}{3}} + \omega^{\frac{1}{3}} + 2, = 2 + a,$$

$$c = \omega^{\frac{7}{3}} + \omega^{\frac{2}{3}}, \quad ,, \quad c^2 = \omega^{\frac{5}{3}} + \omega^{\frac{4}{3}} + 2, = 2 + b.$$

The equation $x^3 - 5x^2 + 6x - 1 = 0$, which, writing therein $x + 2$ for x, gives $x^3 + x^2 - 2x - 1 = 0$ is considered in Hermite's *Cours d'Analyse*, Paris 1873, p. 12, and this suggested to me the foregoing investigation.

NOTE ON MR. KLEIBER'S FUNCTIONS K_i AND G_i.

By *J. W. L. Glaisher*.

§ 1. THE expansions of K and G in ascending powers of $h' - h$ given by Mr. Kleiber in § 29 of his paper, (pp. 29, 30 of this volume) do not agree with those given in Vol. XIX., pp. 146–150 (February, 1890), and it is easy to see that the former are incorrect. I proceed therefore to investigate the expansions of K_i and G_i in powers of $h' - h$.

The function K_i, §§ 2–6.

§ 2. Mr. Kleiber's identification of K and W with $P_{-\frac{1}{2}}$ and $P_{\frac{1}{2}}$ in § 11 (pp. 10, 11) is not very precise; but we may regard K_i as defined by equation (146), p. 27, viz.

$$\frac{2K_i}{\pi} = 1 - \frac{i^2 - \frac{1}{4}}{1^2} h + \frac{(i^2 - \frac{1}{4})(i^2 - \frac{9}{4})}{1^2 . 2^2} h^2 - \frac{(i^2 - \frac{1}{4})(i^2 - \frac{9}{4})(i^2 - \frac{25}{4})}{1^2 . 2^2 . 3^2} h^3 + \&c.$$

We know also that K_i satisfies the differential equation (115), viz.

$$hh' \frac{d^2 u}{dh^2} + (h - h') \frac{du}{dh} + (i^2 - \tfrac{1}{4}) u = 0;$$

and we have also

$$K_0 = K \text{ and } K_1 = 2W = I + E.$$