then $x/(N-x)$ can be expressed as a continued fraction of the form

$$\frac{1}{a_0-1}+\frac{1}{a_1}+\ldots+\frac{1}{a_{n-1}}+\frac{1}{a_n}+\frac{1}{a_{n-1}}+\ldots+\frac{1}{a_1}+\frac{1}{a_0};$$

and N is equal to the continuant

$$(a_0, a_1, \ldots, a_{n-1}, a_n, a_{n-1}, \ldots, a_1, a_0),$$

of which $(a_0, a_1, \ldots, a_{n-1})$ is a factor.

In Mersenne's problem, $N=100895598169$. If we take $y=32$ and $x=1796847$, then $x^2=Ny+1$, and we find

$$\frac{x}{N-x}=\frac{1796847}{100893801322}=\frac{1}{56150}+\frac{1}{2}+\frac{1}{7}+\frac{1}{2}+\frac{1}{56151},$$

and $$N=(56151,\ 2,\ 7,\ 2,\ 56151).$$

Hence a factor of N is the continuant (56151, 2), which is equal to

$$\begin{vmatrix} 56151, & 1 \\ -1\ , & 2 \end{vmatrix}$$

that is, 112303. Hence a factor of N is known.

The question proposed to Fermat and his answer seem to have escaped the attention of many writers on the theory of numbers; but, as far as I know, no solution of the problem has been hitherto published, and therefore it is particularly interesting to find that it is covered by the theorem given by Mr. Birch.

September 22, 1892.

NOTE ON PSEUDO-ELLIPTIC INTEGRALS.

By *W. Burnside.*

Any integral which while apparently elliptic may really be reducible to a logarithm is expressible as the sum of a number of terms of the form

$$\int \frac{(x-a)\,dx}{(x-b)\sqrt{\{f_4(x)\}}},$$

where $f_4(x)$ is a rational integral quartic function. For the case in which the integral consists of a single term I propose

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to determine explicitly, from the elliptic-function point of view, its most general form.

If g_2, g_3 are the two invariants of the quartic $f_4(x)$ the above integral can always by a linear substitution be transformed into

$$\int \frac{(z-\alpha)\,dz}{(z-\beta)\sqrt{(4z^3-g_2z-g_3)}},$$

or if

$$z=P(u), \text{ into } \int \frac{P(u)-P(w)}{P(u)-P(v)}\,du.$$

Introducing the ζ and σ functions, the indefinite integral may be expressed in the form

$$u+\frac{P(v)-P(w)}{P'(v)}\left[2u\,\zeta(v)+\log\frac{\sigma(u-v)}{\sigma(u+v)}\right].$$

If now $2\omega, 2\omega'$ is any pair of primitive periods of the elliptic-function $P(u)$, and 2η, $2\eta'$ the corresponding quasi-periods of $\zeta(u)$; the two elliptic-periods of the integral, or the two constants by which the integral increases when the original variable describes two independent period-paths, are obtained by writing 2ω and $2\omega'$ for u in the above indefinite integral.

The periods so obtained are

$$2\omega+\frac{P(v)-P(w)}{P'(v)}[4\omega\,\zeta(v)-4\eta v],$$

and

$$2\omega'+\frac{P(v)-P(w)}{P'(v)}[4\omega'\zeta(v)-4\eta' v].$$

Besides these two periods the integral has a third by which it increases when the original variable describes a closed path surrounding either of the points at which the integral is infinite.

If $u=v+v'$, the integral expanded in ascending powers of v' becomes

$$\frac{P(v)-P(w)}{P'(v)}\int\frac{dv'}{v'}+\text{finite terms},$$

and hence the value of the integral round a closed path surrounding $v'=0$, *i.e.* the logarithmic period, is

$$2\pi i\,\frac{P(v)-P(w)}{P'(v)}.$$

If now the integral be pseudo-elliptic it must have one

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period only, and the two elliptic-periods must be multiples or sub-multiples of the logarithmic period.

Hence m, m' and n being suitably chosen positive or negative integers without a common factor, the following two equations are thus obtained, viz.

$$\omega\left[\frac{P'(v)}{P(v)-P(w)}+2\zeta(v)\right]-2\eta v=-\frac{m'}{n}\pi i,$$

$$\omega'\left[\frac{P'(v)}{P(v)-P(w)}+2\zeta(v)\right]-2\eta' v=\frac{m}{n}\pi i.$$

Multiplying the first of these by ω' and the second by ω, and subtracting

$$v=\frac{m\omega+m'\omega'}{n};$$

and multiplying by η', η, and subtracting

$$\frac{P'(v)}{P(v)-P(w)}=2\left[\frac{m\eta+m'\eta'}{n}-\zeta(v)\right].$$

The constant v may therefore be any sub-multiple of a period (a result which is well-known), and when it has so been chosen $P(w)$ is determined by the last written equation. It remains to shew that regarding $P(v)$ as known, the determination of $P(w)$ so obtained is algebraical and rational.

For this purpose it is convenient to introduce the function that M. Halphen denotes by $\psi_n(u)$.

When n is odd $\psi_n(u)$ is a rational integral function of $P(u)$ with coefficients rational and integral in g_2 and g_3: and when n is even it is the product of $P'(u)$ by such a function.

The function $\psi_n(u)$ is connected with the σ-function by the equation

$$\frac{\sigma(nu)}{\{\sigma(u)\}^{n^2}}=\psi_n(u),$$

and therefore differentiating logarithmically

$$\frac{1}{n}\zeta(nu)-\zeta(u)=\frac{1}{n^2}\frac{\psi_n'(u)}{\psi_n(u)}.$$

If now v has been chosen so that m and m' are not both even,

$$\zeta(nv)=\zeta(m\omega+m\omega'')=m\eta+m'\eta',$$

and
$$\frac{m\eta+m'\eta'}{n}-\zeta(v)=\frac{1}{n^2}\frac{\psi'_n(v)}{\psi_n(v)},$$

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and therefore, $\psi_n(v)$ or $\psi'_n(v)$ containing $P'(v)$ as a factor, $P(w)$ is in this case at once determined rationally in terms of $P(v)$.

If on the other hand m and m' are both even, n being therefore odd and v an odd sub-multiple of a complete period, $\zeta(nv)$ is infinite and $P(w)$ must be otherwise found.

In this case, since

$$(n-1)\,v = m\omega + m'\omega' - v \ (m, m' \text{ both even});$$

therefore $$\zeta\{(n-1\,.\,v)\} + \zeta(v) = m\eta + m'\eta';$$

but $$\zeta\{(n-1\,.\,v)\} - (n-1)\,\zeta(v) = \frac{1}{n-1}\,\frac{\psi'_{n-1}(v)}{\psi_{n-1}(v)},$$

and therefore $$\frac{m\eta + m'\eta'}{n} - \zeta(v) = \frac{1}{n(n-1)}\,\frac{\psi'_{n-1}(v)}{\psi'_{n-1}(v)}.$$

The results obtained may be summarised thus.

In order that the integral

$$\int\left(\frac{1}{P(u) - P(v)} + A\right) du,$$

where A is a constant, may be pseudo-elliptic v must be a sub-multiple of a period.

If v is an even sub-multiple, so that

$$2mv = 2\bar{\omega},$$

then $$A = \frac{2}{m^2}\,\frac{\psi'_m(v)}{P'(v)\,\psi_m(v)},$$

and if v is an odd submultiple, so that

$$(2m+1)\,v = 2\bar{\omega},$$

then $$A = \frac{1}{m(2m+1)}\,\frac{\psi_{2m}'(v)}{P'(v)\,\psi_{2m}(v)}.$$

If v is a half-period, $P'(v)$ is zero, and therefore the previous reasoning does not apply. But in this case the integral is one of the second species, and not of the third; and its being pseudo-elliptic is therefore out of the question.

Writing now generally

$$v = \frac{2\bar{\omega}}{n},$$

where n may be odd or even, and denoting by $2\bar{\eta}$ the quasi-

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period of ζ corresponding to $2\bar{\omega}$, the previously given value of the integral

$$u+\frac{P(v)-P(w)}{P'(u)}\left[2u\,\zeta(v)+\log\frac{\sigma(u-v)}{\sigma(u+v)}\right]$$

becomes $$\frac{P(v)-P(w)}{P'(v)}\left[\frac{4\bar{\eta}u}{n}+\log\frac{\sigma\left(u-\frac{2\bar{\omega}}{n}\right)}{\sigma\left(u+\frac{2\bar{\omega}}{n}\right)}\right],$$

and therefore

$$\int\left[\frac{1}{P(u)-P\left(\frac{2\bar{\omega}}{n}\right)}+A\right]du=\frac{1}{P'\left(\frac{2\bar{\omega}}{n}\right)}\log\frac{e^{\frac{2\bar{\eta}u}{n}}\sigma\left(u-\frac{2\bar{\omega}}{n}\right)}{e^{-\frac{2\bar{\eta}u}{n}}\sigma\left(u+\frac{2\bar{\omega}}{n}\right)}.$$

The n^{th} power of the quantity under the logarithm is unaltered when u is increased by a complete period, and is therefore a rational function of $P(u)$ and $P'(u)$; its numerator may be shewn to be equal to (Halphen, I., p. 223)

$$\begin{vmatrix} 1, & P(u), & P'(u), & P''(u), & \ldots, & P^{(n-2)}(u) \\ 1, & P(v), & P'(v), & \ldots\ldots & \ldots, & P^{(n-2)}(v) \\ 0, & P'(v), & \ldots\ldots & & \ldots, & P^{(n-1)}(v) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & P^{(n-2)}(v), & \ldots\ldots & & , & P^{(2n-4)}(v) \end{vmatrix},$$

while the denominator only differs from the numerator in the signs of the odd differential coefficients of $P(u)$.

With increasing values of n these expressions rapidly become very complicated, but the algebraical form of the relation in the case when $n=3$ is sufficiently simple to be worth giving explicitly.

Thus, if $$v=\tfrac{2}{3}\bar{\omega},$$
then $P(v)=\alpha$ is given by

$$\alpha^4-\tfrac{1}{2}g_2\alpha^2-g_3\alpha-\tfrac{1}{48}g_2^{\,3}=0,$$

which can also be written in the form

$$12P(v)\,P'^2(v)=P''^2(v).$$

Hence $$A=\tfrac{1}{3}\frac{\psi_3'(v)}{P'(v)\,\psi_3(v)}=\tfrac{1}{3}\frac{P''(v)}{P'^2(v)}=\frac{4P(v)}{P''(v)},$$

and $$\frac{1}{P(u)-P(v)}+A=\frac{4P(v)}{P''(v)}\,\frac{P(u)+\tfrac{1}{2}\alpha-\frac{g_2}{8\alpha}}{P(u)-\alpha}.$$

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Therefore

$$\int \frac{P(u)+\frac{1}{2}\alpha-\frac{g_2}{8\alpha}}{P(u)-\alpha}\,du=\frac{P''(v)}{4P(v)\,P'(v)}\log\frac{e^{\frac{2}{3}\eta u}\,\sigma\,(u-\frac{2}{3}\omega)}{e^{-\frac{2}{3}\eta u}\,\sigma\,(u+\frac{2}{3}\omega)}$$

$$=\frac{P'(v)}{P''(v)}\log\frac{e^{2\eta u}\,\sigma^3(u-\frac{2}{3}\omega)}{e^{-2\eta u}\,\sigma^3(u+\frac{2}{3}\omega)}$$

$$=\frac{P'(v)}{P''(v)}\log\frac{P''(v)\,P(u)+P'^2(v)-P(v)\,P''(v)-P'(u)\,P'(v)}{P''(v)\,P(u)+P'^2(v)-P(v)\,P''(v)+P'(u)\,P'(v)}.$$

If now $P(u)=z,\ P'(u)=-\sqrt{(4z^3-g_2z-g_3)}$,

the algebraical form of the equation is

$$\int \frac{\left(z+\frac{1}{2}\alpha-\frac{g_2}{8\alpha}\right)dz}{(z-\alpha)\sqrt{(4z^3-g_2z-g_3)}}$$

$$=\frac{\sqrt{(4\alpha^3-g_2\alpha-g_3)}}{6\alpha^2-\frac{1}{2}g_2}$$

$$\times\log\frac{(6\alpha^2-\frac{1}{2}g_2)z-2\alpha^3-\frac{1}{2}g_2\alpha-g_3-\sqrt{(4\alpha^3-g_2\alpha-g_3)}\sqrt{(4z^3-g_2z-g_3)}}{(6\alpha^2-\frac{1}{2}g_2)z-2\alpha^3-\frac{1}{2}g_2\alpha-g_3+\sqrt{(4\alpha^3-g_2\alpha-g_2)}\sqrt{(4z^3-g_2z-g_3)}}.$$

The next case of $n=4$, which when written with complete generality takes a far from simple form in Weierstrass's notation, is the simplest of all if the general elliptic differential be dealt with.

Thus, if $\dfrac{y-p}{y-q}=-\dfrac{x-p}{x-q}$

be one of the three linear substitutions which transform

$$\frac{dx}{\sqrt{(x-\alpha.x-\beta.x-\gamma.x-\delta)}}$$

into itself, then

$$I=\int\frac{x-p}{x-q}\,\frac{dx}{\sqrt{(x-\alpha.x-\beta.x-\gamma.x-\delta)}}$$

is pseudo-elliptic.

Suppose that α, β and γ, δ are respectively interchanged by the substitution in question; and write

$$z=\frac{x-p}{x-q}.$$

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Then $z - z_{\alpha} = \frac{(x-\alpha)(p-q)}{(x-q)(\alpha-q)}$, $z - z_{\gamma} = \frac{(x-\gamma)(p-q)}{(x-q)(\gamma-q)}$,

$$z - z_{\beta} = z + z_{\alpha} = \frac{(x-\beta)(p-q)}{(x-q)(\beta-q)}, \quad z + z_{\gamma} = \frac{(x-\delta)(p-q)}{(x-q)(\delta-q)};$$

and therefore

$$I = \frac{p-q}{\sqrt{(\alpha - q\,.\,\beta - q\,.\,\gamma - q\,.\,\delta - q)}} \int \frac{zdz}{\sqrt{(z^2 - z_{\alpha}^{\ 2}\,.\,z^2 - z_{\gamma}^{\ 2})}},$$

which is pseudo-elliptic.

This case clearly corresponds to the two obviously pseudo-elliptic integrals

$$\int \frac{xdx}{\sqrt{(1-x^2\,.\,1-k^2x^2)}} \text{ and } \int \frac{dx}{x\sqrt{(1-x^2\,.\,1-k^2x^2)}},$$

which immediately offer themselves when Jacobi's normal form is used. The other four forms in this case are

$$\int \left[\frac{1 - x\sqrt{k}}{1 + x\sqrt{k}} \text{ or } \frac{1 + x\sqrt{k}}{1 - x\sqrt{k}}\right] \frac{dx}{\sqrt{(1-x^2\,.\,1-k^2x^2)}},$$

and $$\int \left[\frac{1 - x\sqrt{(-k)}}{1 + x\sqrt{(-k)}} \text{ or } \frac{1 + x\sqrt{(-k)}}{1 - x\sqrt{(-k)}}\right] \frac{dx}{\sqrt{(1-x^2\,.\,1-k^2x^2)}}.$$

ON THE DIVISION OF THE PERIODS OF ELLIPTIC FUNCTIONS BY 9.

By *W. Burnside, M.A.*

The equation which determines $P(3u)$ in terms of $P(u)$ is

$$(3P^4 - \tfrac{3}{2}g_2P^2 - 3g_3P - \tfrac{1}{16}g_2^{\ 2})^2\,P(3u)$$
$$= P^9 + 3g_2P^7 + 24g_3P^6 + \tfrac{15}{8}g_2^{\ 2}P^5 - \tfrac{2}{3}g_2g_3P^4 + (3g_3^{\ 2} - \tfrac{9}{16}g_2^{\ 3})\,P^3$$
$$+ (\tfrac{9}{256}g_2^{\ 4} - \tfrac{3}{2}g_2g_3^{\ 2})\,P + \tfrac{1}{32}g_2^{\ 3}g_3 - g_3^{\ 3} \ \ldots\ldots\ldots\ldots \text{(i)},$$

where for brevity P is written for $P(u)$.

If $$P(3u) = P(3u_0),$$

the nine roots of this equation are

$$\begin{array}{lll} P(u_0), & P(u_0 + \tfrac{2}{3}\omega), & P(u_0 + \tfrac{4}{3}\omega), \\ P(u_0 + \tfrac{2}{3}\omega'), & P(u_0 + \tfrac{2}{3}\omega' + \tfrac{2}{3}\omega), & P(u_0 + \tfrac{2}{3}\omega' + \tfrac{4}{3}\omega), \\ P(u_0 + \tfrac{4}{3}\omega'), & P(u_0 + \tfrac{4}{3}\omega' + \tfrac{2}{3}\omega), & P(u_0 + \tfrac{4}{3}\omega' + \tfrac{4}{3}\omega), \end{array}$$

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