

divisible by  $1.2 \dots r$ ; for then, writing  $Nx$  for  $x$ , the form of the coefficient would have been

$$\frac{KN^r}{n^r \cdot n^{\frac{1}{2}r \cdot r-1}}, = \frac{KN^r}{n^{\frac{1}{2}r \cdot r+1}},$$

and there would be no value (however great) of  $N$  by which the denominator factor  $n^{\frac{1}{2}r \cdot r+1}$  could be got rid of.

### NOTE ON LINEAR SUBSTITUTIONS.

By *W. Burnside.*

THE points  $z_1, z_2$  which are unchanged by a linear substitution

$$w = \frac{\alpha z + \beta}{\gamma z + \delta} \quad (\alpha\delta - \beta\gamma = 1) \dots\dots\dots(i)$$

are the roots of

$$\gamma z^2 + (\delta - \alpha)z - \beta = 0,$$

and are therefore given by

$$\gamma (z_1 \text{ or } z_2) = \frac{1}{2}(\alpha - \delta) \pm \frac{1}{2}\sqrt{(\alpha + \delta)^2 - 4}.$$

When  $\alpha + \delta$  is not = 2, equation (i) may be written in the form

$$\frac{w - z_1}{w - z_2} = K \frac{z - z_1}{z - z_2} \dots\dots\dots(ii),$$

and  $K$ , the multiplier, is given by

$$K + \frac{1}{K} = (\alpha + \delta)^2 - 2.$$

When  $\alpha + \delta = 2$ , so that the fixed points  $z_1, z_2$  coincide, the form (ii) becomes illusory, and we have instead

$$\frac{1}{w - z_1} = \frac{1}{z - z_1} + c \dots\dots\dots(iii).$$

When the substitution takes this latter form it is called parabolic: when it is of the form (ii) it is called hyperbolic, elliptic or loxodromic, according as  $K$  is real, imaginary with modulus unity, or imaginary with modulus other than unity.

It is the object of this note to determine the conditions under which two substitutions, neither of which is loxodromic, when performed successively on a variable leads to a substitution which also is not loxodromic.

The first of the two substitutions may, without loss of

generality, be taken with its fixed points at 0 and  $\infty$ ; or if parabolic with its fixed point at infinity and its additive constant real.

Let this be either

$$w = n^2 z \dots\dots\dots(iv)$$

or

$$w = z + m \dots\dots\dots(iv)'$$

The result of performing first (iv) and then (i) is

$$w = \frac{anz + \frac{\beta}{n}}{\gamma nz + \frac{\delta}{n}}$$

and the multiplier of this substitution is given by

$$K + \frac{1}{K} = \left( an + \frac{\delta}{n} \right)^2 - 2,$$

and if  $K$  is either to be real or to have modulus unity

$$an + \frac{\delta}{n} \text{ must be real.}$$

Also if (i) is not loxodromic  $\alpha + \delta$  is real.

Suppose then that

$$\alpha = a + ci, \quad \delta = b - ci.$$

Then (a) when  $n$  is real

$$c \left( n - \frac{1}{n} \right) = 0;$$

(b) when  $n = e^{i\theta}$

$$(a + ci) e^{i\theta} + (b - ci) e^{i\theta} \text{ is real,}$$

or

$$(a - b) \sin \theta = 0.$$

Hence in the first case  $c = 0$  or  $\alpha$  and  $\delta$  are both real, and in the second case  $a = b$  or  $\alpha$  and  $\delta$  are conjugate imaginaries. The result of performing first (iv)' and then (i) is

$$w = \frac{\alpha z + \alpha m + \beta}{\gamma z + \gamma m + \delta},$$

and if in this case (c) the substitution is not loxodromic  $\alpha + \delta + \gamma m$  must be real, and therefore  $\gamma$  must be real.

All possible cases are thus exhausted. Now in case (a) when (i) is hyperbolic the fixed points of (i) lie on a line through the origin, since  $\gamma z_1$  and  $\gamma z_2$  are both real. Hence in this case the two given substitutions being both hyperbolic

their four fixed points lie on a line, and if the hyperbolic substitution (iv) be replaced by one with both its fixed points at finite distance the corresponding condition would be that the four fixed points should lie on a circle. In case (a) when (i) is elliptic the line joining its fixed points is bisected at right-angles by a line through the origin. In this case the substitutions are respectively elliptic and hyperbolic, and the corresponding general condition is that the fixed points of the elliptic substitution should be inverse points in the respect to some circle through the fixed points of the hyperbolic substitution. Case (a) when (i) is parabolic may be regarded as the limit of either of the above cases as the fixed points approach each other indefinitely. In the first of these cases the circle through the four fixed points, and in the second the circle through the two fixed points of the hyperbolic substitution with respect to which the other two fixed points are inverses, are unchanged circles for both the given substitutions; and hence in each of these cases the condition that the resulting substitutions should not be loxodromic is that there should be one circle (at least) which is unchanged by each of the given substitutions. A similarly detailed consideration of cases (b) and (c) leads to the same condition. Hence in every case it is a necessary condition that there should be one circle unchanged by both the given substitutions. From the nature of a loxodromic substitution which leaves no circle unchanged, it is obvious that this is a sufficient condition.

It follows almost immediately that the group of substitutions arising from any given set of fundamental substitutions will or will not contain loxodromic substitutions according as there is not or is at least one circle which is unchanged by the given set; and that if such a group contains no loxodromic substitution there must be a circle which is unchanged by the set of fundamental substitutions.

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CORRECTION TO THE NOTE ON THE NINE SCHOOLGIRLS  
PROBLEM, (p. 159).

THE first paragraph of the Note should run:

'This is a parallel question to the well-known one of fifteen schoolgirls extended to the supposition of their walking for one week, three and three together, so that in any the same day no two, and at the end of the week no three, taking four walks a day, shall have walked more than once together.'

END OF VOL XXII.

