MESSENGER OF MATHEMATICS.

NUMBER OF PROPER TERNARY n-ICS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., London.

1. *Introduction.* A QUANTIC of n^{th} degree in x, y may be said to be *complete* when all the possible terms are present, and to be *incomplete* when some of the possible terms are absent, so that it contains a number of terms r less than the full number of terms of the complete quantic.

A quantic of n^{th} degree may be said to be a *proper* quantic when it is not the product of algebraic factors of lower degrees. A quantic which contains a linear or other algebraic factor may be styled an *improper* quantic.

It is proposed to investigate in this Paper the number of *incomplete proper ternary n^{th} degree* quantics, arising from the complete ternary n^{th} degree quantic by erasure of some of its terms. This number is interesting as being the number of proper n-ic equations, and also the number of differential equations arising from an n-ic equation.

From the definitions it follows that:

(1) Every complete ternary quantic is (in general) a *proper* ternary quantic.

(2) Every binary quantic is an *improper* quantic.

2. *Preliminary Formulæ*:—

Let n be the degree of a function in x, y.

Let u_n, U_n be the types of binary and ternary quantic for n^{th} degree, where

$$u_n = a_{n,0}x^n + a_{n-1,1}x^{n-1}y + a_{n-2,2}x^{n-2}y^2 + \ldots + a_{1,n-1}xy^{n-1} + a_{0,n}y^n \quad\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$U_n = u_n + u_{n-1} + u_{n-2} + \ldots + u_2 + u_1 + u_0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

so that $u_0 = c$ (a constant) ....(3).

Let t_n, T_n = number of terms in u_n, U_n respectively; therefore $t_n = (n+1)$;

$$T_n = \{(n+1) + n + (n-1) + \ldots + 3 + 2 + 1\} = \tfrac{1}{2}(n+1)(n+2) \quad \ldots\ldots\ldots\ldots\ldots(4).$$

Let $C(n, r)$ = number of combinations of n different things taken r together.

$$= n!/r!(n-r)! \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Let $S(n, r)$ = number of sets, containing r terms, formable from U_n.

Let $s(n, r)$ = number of sets, containing r terms, formable from U_n, each set containing at least one term from u_n.

Let $\Sigma(n, r)$ = number of functions of n^{th} degree, containing r terms, (including improper quantics) formable from U_n.

Let $\sigma(n, r)$ = number of improper quantics of n^{th} degree, containing r terms, formable from U_n.

Let $N'''(n, r)$ = number of proper ternary quantics of n^{th} degree, containing r terms, formable from U_n.

Let $N'''(n)$ = total number of proper ternary quantics of n^{th} degree formable from U_n.

Thus $N'''(n, r)$ and $N'''(n)$ are the numbers sought.

[The accents in the symbols $N'''(n, r)$, $N'''(n)$ indicate that these symbols refer to *ternary* quantics: this distinction is required for use in a subsequent Paper (p. 8) on *quaternary* quantics.]

$$\text{Then } S(n, r) = C(T_n, r) = T_n!/r!(T_n - r)! \quad \ldots\ldots\ldots(6),$$

$$s(n, r) = S(n, r) - S\{(n-1), r\} \quad \ldots\ldots\ldots\ldots\ldots(7),$$

$$= C(T_n, r) - C(T_{n-1}, r) \quad \ldots\ldots\ldots\ldots\ldots(7a),$$

$$\Sigma(n, r) = s(n, r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$N'''(n, r) = \Sigma(n, r) - \sigma(n, r) \quad \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$$= s(n, r) - \sigma(n, r) \quad \ldots\ldots\ldots\ldots\ldots\ldots(9a).$$

The computation of $\sigma(n, r)$ will occupy most of the rest of this paper, and in fact presents the only difficulty.

3. *Decomposition of* $\sigma(n, r)$ *into parts.*

A ternary n-ic is an *improper* quantic in following cases:

I. When containing x or y, or both x and y as factors.

II. When it is a function of x only (not of y), or of y only (not of x).

III. When it is a homogeneous function of n^{th} degree (a binary n-ic).

These cases are to a considerable extent mutually involved, *e.g.*

$(a_{n,0}x^n + a_{m,0}x^m + a_{p,0}x^p)$ falls under both Cases I, II,

$(a_{n,0}x^n + a_{n-m,m}x^{n-m}y^m + a_{n-p,p}x^{n-p}y^p)$ falls under both Cases I, III,

and the only difficulty consists in avoiding counting such cases twice.

Let $\sigma(n, r, x, y, xy) =$ the number of n-ic functions of r terms, containing x or y, or both x and y as a factor, formable from U_n.

Let $\sigma(n, r, fx, fy) =$ the number of n-ic functions of r terms, which are functions of x only (not of y), or of y only (not of x), but not containing either x or y as a factor, formable from U_n.

Let $\sigma(n, r, u_n) =$ the number of homogeneous n-ic functions of r terms, not containing x or y as a factor, nor functions of x only or of y only.

$$\text{Then } \sigma(n, r) = \sigma(n, r, x, y, xy) + \sigma(n, r, fx, fy) + \sigma(n, r, u_n) \quad \text{............(10).}$$

Thus the three parts of $\sigma(n, r)$ are herein defined so as to exclude twice counting of functions falling under two of the Cases I, II, III.

It remains now to compute the three parts of $\sigma(n, r)$.

4. *Number of terms containing x, y.*

Let ξ_n, X_n be the number of terms containing x in u_n, U_n.

Let η, Y_n be the " " " y in u_n, U_n.

Let λ_n, L_n be the " " x and y in u_n, U_n.

$$\text{Then } \xi_n = n = \eta_n;\ \lambda_n = n - 1 \quad \text{..........................(11).}$$

$$\text{Then } X_n = \Sigma(\xi_n) = \{n + (n-1) + \ldots + 3 + 2 + 1\}$$
$$= \tfrac{1}{2}n(n+1) = Y_n = T_{n-1} \quad \text{.........(12).}$$

$$\text{Then } L_n = \Sigma(\lambda_n) = \{(n-1) + (n-2) + \ldots + 3 + 2 + 1\}$$
$$= \tfrac{1}{2}n(n-1) = X_{n-1} = T_{n-2} \quad \text{....(13).}$$

5. *Computation of $\sigma(n, r, x, y, xy)$.* This may be decomposed into the algebraic sum of *three* parts, viz.

2 parts, when only one of x, y enter as factors.

1 part, when both x, y enter as factors.

Let $S(n, r, x)$, $S(n, r, y)$, $S(n, r, xy)$ be the numbers of sets of r terms, containing x or y, or both x and y as factors, formable from the X_n, Y_n, or L_n terms of U_n, which contain x or y, or both x and y as factors in U_n respectively.

Let $s(n, r, x)$, $s(n, r, y)$, $s(n, r, xy)$ be the numbers of sets of r terms, containing x or y, or both x and y as factors, formable from the X_n, Y_n, or L_n terms of U_n, which contain x or y, or both x and y as factors; each set containing at least one term from u_n.

Let $\sigma(n, r, x)$, $\sigma(n, r, y)$, $\sigma(n, r, xy)$ be the number of n-ic functions of r terms, containing x or y, or both x and y as factors, formable from U_n.

Then

$$S(n, r, x) = C(X_n, r) = C(Y_n, r) = S(n, r, y) \quad \text{.........(14)},$$

$$s(n, r, x) = S(n, r, x) - S\{(n-1), r, x\} = s(n, r, y) \quad \text{..(15)},$$

$$\sigma(n, r, x) = s(n, r, x) = s(n, r, y) = \sigma(n, r, y) \quad \text{......(16)},$$

$$= C(X_n, r) - C(X_{n-1}, r) \quad \text{.....................(16a)}.$$

$$S(n, r, xy) = C(L_n, r) \quad \text{.....................................(17)},$$

$$s(n, r, xy) = S(n, r, xy) - S\{(n-1), r, xy\} \quad \text{...........(18)},$$

$$\sigma(n, r, xy) = s(n, r, xy) \quad \text{....................................(19)},$$

$$= C(L_n, r) - C(L_{n-1}, r) \quad \text{.....................(19a)}.$$

And as the whole of the sets $\sigma(n, r, xy)$ are clearly included in both the numbers $\sigma(n, r, x)$, $\sigma(n, r, y)$, therefore

$$\sigma(n, r, x, y, xy) = \sigma(n, r, x) + \sigma(n, r, y) - \sigma(n, r, xy) \quad \text{...(20)},$$

$$= 2\{C(X_n, r) - C(X_{n-1}, r)\} - \{C(L_n, r) - C(L_{n-1}, r)\} \quad \text{...(21)},$$

$$= 2C(T_{n-1}, r) - 3C(T_{n-2}, r) + C(T_{n-3}, r) \quad \text{............(21a)}.$$

6. *Number of terms containing only one variable.*

Let X_n', Y_n' be the number of terms of form $a_{m,0}x^m$ or $a_{0,m}y^m$ contained in U_n, where m takes all the values 1, 2, 3, ..., n (but not zero).

Then $$X_n' = n = Y_n' \quad \text{....................(22)}.$$

7. *Computation of* $\sigma(nr, fx, fy)$. This may be decomposed into the sum of two parts as follows:

Let $\sigma(n, r, fx)$, $\sigma(n, r, fy)$ be the number of n-ic functions of r terms formable from U_n, which are functions of x only or of y only, and yet not containing either x or y as a factor.

It is clear that each set included in these numbers must contain the absolute term u_0 $(=c)$, and $(r-1)$ other terms taken from the X_n' or Y_n' terms of Art. 6 of form $a_{m,0}x^m$ or $a_{0,m}y^m$; therefore

$$\sigma(n, r, fx) = \sigma(n, r, fy) = C(X'_n - 1, r-2)$$
$$= C(n-1, r-2) \quad \text{............(23)}.$$

Hence, as the sets included in $\sigma(n, r, fx)$, $\sigma(n, r, fy)$ are wholly different,

$$\sigma(n, r, fx, fy) = \sigma(n, r, fx) + \sigma(n, r, fy)$$
$$= 2C\{(n-1), (r-2)\} \quad \text{....................(23}a\text{)}.$$

8. *Computation of* $\sigma(n, r, u_n)$. By definition (Art. 3) this is the number of homogeneous quantics of r terms of n^{th} degree, not containing x or y as factors, nor yet functions of x only or of y only.

It is clear that all the quantics included in this last number σ must be *binary* quantics, and must all contain the two terms $a_{n,0}x^n + a_{0,n}y^n$ and $(r-2)$ other terms taken from the remaining $(n-1)$ terms of u_n.

Therefore $\sigma(n, r, u_n) = C\{(n-1), (r-2)\}$ ....(24),

9. *Reduction of* $\sigma(n, r)$. Combining the different parts of $\sigma(n, r)$ by Results (10), (21a), (23a), (24),

$$\sigma(n, r) = [2C(T_{n-1}, r) - 3C(T_{n-2}, r) + C(T_{n-3}, r)]$$
$$+ 3C\{(n-1), (r-2)\} \quad \text{.........(28*)}.$$

10. *Final formula for* $N'''(n, r)$. By (9a), (7a), (28), the number sought is—

$$N'''(n, r) = [C(T_n, r) - 3C(T_{n-1}, r) + 3C(T_{n-2}, r)$$
$$- C(T_{n-3}, r)] - 3C\{(n-1), (r-2)\} \text{......(29)},$$

wherein, by (4)

$$T_n = \tfrac{1}{2}(n+1)(n+2),\ T_{n-1} = \tfrac{1}{2}n(n+1),\ T_{n-2} = \tfrac{1}{2}n(n-1),$$
$$T_{n-3} = \tfrac{1}{2}(n-1)(n-2) \text{.....................(30)}.$$

Hence, from the meaning of C, or from first principles,

r not < 2, nor $> T_n$ *i.e.* not $> \frac{1}{2}(n+1)(n+2)$...(31).

---

* There is a hiatus here in the numbering of the Results, (Nos. 25 to 27 being omitted): this is to allow of the numbering of the Results in the subsequent Paper (similar to this) on the "Number of Proper Quaternary n-ics," agreeing with the numbering of similar Results in this Paper.

Also, the greater the value of r, the simpler the general expression (29) for $N'''(n, r)$ becomes, because

$$C\{(n-1),\ (r-2)\}=0,\quad \text{when } r>(n+1)\ \ldots\ldots(32),$$

$$C(T_{n-3}, r)=0,\quad \text{when } r>T_{n-3}\ \ldots\ldots\ldots(32a),$$

$$C(T_{n-3}, r)=0,\quad \text{when } r>T_{n-2}\ \ldots\ldots\ldots(32b),$$

$$C(T_{n-1}, r)=0,\quad \text{when } r>T_{n-1}\ \ldots\ldots\ldots(32c).$$

Hence, the following Table of values of $N'''(n, r)$ for different values of r.

Value of r.	General expressions for $N'''(n, r)$.	Result. No.
$=2$	$3(n-1)$	(33a)
$=3$	$\frac{1}{2}\{5n^3-9n+6\}$	(33b)
$=4$	$\frac{1}{8}(n-1)\{n(n+1)(7n^2-8)-12(n-2)\}$	(33c)
⋮		......
$<(n+1)$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)-C(T_{n-3}, r)]$ $-3C\{(n-1), (r-2)\}$	(33d)
$=(n+1)$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)-C(T_{n-3}, r)]-3$	(33e)
$<T_{n-3}$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)-C(T_{n-3}, r)]$ $-3C\{(n-1), (r-2)\}$	(33f)
$=T_{n-3}$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)-1]$ $-3C\{(n-1), (r-2)\}$	(33g)
$>T_{n-3},\ <T_{n-2}$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)]-3C\{(n-1), (r-2)\}$	(33h)
$=T_{n-2}$	$[C(T_n, r)-3C(T_{n-1}, r)+3]-3C\{(n-1), (r-2)\}$	(33i)
$>T_{n-2},\ <T_{n-1}$	$[C(T_n, r)-3C(T_{n-1}, r)]-3C\{(n-1), (r-2)\}$	(33j)
$=T_{n-1}$	$[C(T_n, r)-3]-3C\{(n-1), (r-2)\}$	(33k)
$>T_{n-1},\ <T_n$	$[C(T_n, r)]$	(33l)
$=T_n$	1	(33m)

Note that the term $C\{(n-1), (r-2)\}$ enters into the *general* formulæ (33d...k) effectively only when n, r are both small, and quickly dissappears as r increases.

11. *Computation of* $N'''(n)$.

From the definitions, Art 2, the final number sought,

$$N'''(n)=\Sigma\{N'''(n, r)\} \text{ from } r=2 \text{ to } r=T_n\ldots\ldots(34).$$

But, by the theory of combinations, (m is any integer $>r$)

$$\Sigma[C(m, r), \text{ from } r=2 \text{ to } r=m]$$

$$=\Sigma[C(m, r), \text{ from } r=1 \text{ to } r=m]-C(m, 1)$$

$$=(2^m-1)-m\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35a),$$

and $\Sigma[C\{(n-1),(r-2)\}$, from $r=2$ to $r=(n+1)]=\Sigma[C\{(n-1),\rho\}$, from $\rho=0$ to $\rho=(n-1)]=2^{n-1}$...(35b);

therefore by results (34), (29), (35a, b) the final formula is $N'''(n)=[2^{T_n}-2^{T_{n-3}})-3(2^{T_{n-1}}-2^{T_{n-2}})]$
$-[(T_n-T_{n-3})-3(T_{n-1}-T_{n-2})]-3\times 2^{n-1}=[(2^{T_n}-2^{T_{n-3}})-3(2^{T_{n-1}}-2^{T_{n-2}})]-3\times 2^{n-1}$ ....(36).

$$=2^{\frac{1}{2}n^2}\times(2^{\frac{3}{2}n}-2^{-\frac{3}{2}n})[2\times(2^n+2^{-n})-1]-3\times 2^{n-1} \text{......(36a)}$$

a remarkable formula.

The following Table shows the values of T_n, $N'''(n, r)$, and $N'''(n)$ for the ternary n-ic quantics defined by $n=1, 2, 3, 4$. It will be seen how very rapidly $N'''(x)$ increases with n.

n	T_n	Value of $N(n, r)$. Values of r.														Value of $N(n)$.
		2	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	3	0	1	...	...	...	...	...	...	...	...	...	...	...	...	1
2	6	3	14	15	6	1	...	...	...	...	...	...	...	...	...	39
3	10	6	57	162	234	207	120	45	10	1	...	...	...	...	...	842
4	15	9	145	771	2,262	4,378	6,075	6,300	4,975	3,000	1,365	455	105	15	1	29,856
5	21	Details not computed; r ranges from 2 to 21.														2,001,808

[To ensure accuracy the values of $N(n)$ have been computed from the general formula (36), and found to tally with those given by adding the values of $N(n, r)$. The values of $N(n, r)$ have also been verified for the cases $n=1$ and 2 completely, and partly for the case $n=3$, (viz. when $r=2$ and 3) by actually writing out the ternary n-ics themselves].