ON THE FINITE DISPLACEMENT OF A RIGID BODY.

By *W. Burnside.*

In a former note in Vol. XIX. of this journal I have given a geometrical construction for the central axis, translation and angle of rotation of the displacement of a rigid body which results from two given displacements successively performed. I propose here to deal with the same problem by a rather different method, which seems to lend itself very readily to the discussion of certain properties of the displacements of a rigid body which have not hitherto, I believe, been noticed. A rotation through two right angles I shall call a half turn.

LEMMA. If AEB, CFD are two parallel lines and if EF is perpendicular to both of them, half turns round AB and CD in succession are equivalent to a translation $2EF$. This is well known and does not need proof.

THEOREM. Let AEB, CFD be any two lines and let EF be perpendicular to both of them. Then half turns round AB and CD in succession are equivalent to a twist of which EF is the central axis, $2EF$ is the translation, and twice the angle between EA and FC the angle of rotation.

For, draw $A'FB'$ parallel to AB. Then by the preceding Lemma, and since displacements are associative, a half turn round AB is equivalent to a translation $2EF$, followed by a half turn about $A'B'$. Now by the known theorem for the composition of finite rotations about intersecting axes successive half turns about $A'B'$ and CD are equivalent to a rotation about EF through twice the angle $A'FC$. Hence successive half turns round AB and CD are equivalent to a translation $2EF$, and a rotation round EF through twice the angle $A'FC$.

Conversely, any displacement of a rigid body can be performed by successive half turns about two lines, one of which may be any line meeting the central axis of the displacement at right angles, while the other is the line into which this is changed by half the given displacement. It may be noticed that the half of a given displacement, *i.e.* that displacement which performed twice produces the given displacement, is not unique. For if $2x$ and 2α are the trans-

lation and rotation of the displacement in its canonical form,
translation x and rotation α, or translation x and rotation $\alpha + \pi$,
both lead on repetition to the given displacement. This want
of uniqueness, however, evidently does not affect the deter-
mination of the axis of the second half turn when that of the
first is given.

Suppose now that AEB, CFD are the central axes of
any two finite displacements, EF being perpendicular to both.
Take PR meeting AB at right angles so that half the twist
about AB brings PR to EF, and QS meeting CD at right
angles so that half the twist about CD brings FE to QS.
Then the successive twists round AB and CD are equivalent
to four successive half turns round PR, EF, EF, and QS.
The two successive half turns round EF give no displacement
at all, and hence the resultant displacement is equivalent to
successive half turns round PR and QS. If now RS is the
shortest distance between these lines, the resultant displace-
ment is, by the above theorem, a twist with RS for its central
axis, $2RS$ for its displacement, and twice the angle between
PR and QS for its rotation. This agrees, as it should do,
with the construction given in my note in Vol. XIX.

It is well-known that any infinitesimal displacement
(or velocity-system) of a rigid body is expressible in an
infinite number of ways as a pair of infinitesial rotations
(or angular velocities) about two lines, one of which may be
chosen quite arbitrarily; and that if one of these lines lies in
a given plane the other passes through a fixed point in that
plane, and conversely. I shall now shew that there is a series
of precisely similar theorems for finite displacements.

LEMMA. A half turn round AB followed or preceded by
any rotation round any line in the plane ABC is equivalent to
a rotation round some line lying in a plane through AB
perpendicular to the plane ABC.

This follows immediately from the construction for the
resultant of two finite rotations round interesting axes and
does not require proof here.

Let now OP be any line and EF the central axis
of a given displacement. From O draw OEA perpen-
dicular to EF, and take $O'FC$ so that the twist is equi-
valent to half turns about OEA and $O'FC$. Suppose that
the plane through OEA perpendicular to the plane POA
meets $O'FC$ in O'. Then a half turn round OEA followed
by a rotation round OP through twice the angle between the
planes OPA and OPO' is equivalent to a rotation round OO',
while a line $O'Q$ through O' can be similarly found such that

this rotation round OO' is equivalent to a half turn round $O'FC$, followed by a suitable rotation round $O'Q$. Hence, a half turns round OEA followed by a certain rotation round OP is equivalent to a half turn round $O'FC$, followed by a rotation round $O'Q$. If now a half turn round OEA be carried out *before* each of these equivalent operations and the rotation round $O'Q$ reversed *after* each of them, it follows that the rotation round OP followed by the rotation round $O'Q$ reversed is equivalent to successive half turns round OEA and $O'EC$, that is, to the given twist. This construction seems to depend on choosing a particular point O on the line OP, but it may easily be shewn that the line $O'Q$ and the magnitudes of the two rotations are independent of the point O. For, if possible, suppose the displacement equivalent also to certain rotations round OP and O_1Q_1 successively. The reversed displacement is given by the rotations reversed round O_1Q_1 and OP successively, and hence certain rotations round O_1Q_1, OP, and $O'Q$, three lines which do not all meet in a point, are equivalent to no displacement at all. But this is impossible.

Suppose now in the construction just given that the point O is fixed, but that OP may be any line through O. The resultant of a half turn round $O'FC$ and the rotation round $O'Q$ is, by the second Lemma, a rotation round some line in the plane through $O'FC$ perpendicular to the plane $QO'C$. But it is also equivalent to a half turn round OEA and a rotation round OP, that is to a rotation round some line through O. Hence, O must lie in the plane through $O'FC$ perpendicular to $QO'C$, or in other words $O'Q$ must lie in the plane through $O'FC$, which is perpendicular to the plane $OO'FC$. Hence, when the axis of the first rotation passes through a given point, that of the second lies in a definite plane (not however, in general, containing the point).

Suppose again that the axis of the first rotation lies in a given plane. Let OEA be that line in the given plane which meets the central axis EF of the twist at right angles, and take $O'FC$ as before. The resultant of a half turn round OEA followed by the first rotation round OP is a rotation round some line in the plane through OEA perpendicular to the given plane. But, as before, the half turn round OEA followed by the rotation round OP is equivalent to a half turn round $O'FC$ followed by a rotation round $O'Q$, that is, to a rotation round some point O' of $O'FC$. Hence, O' must be the point in which the line $O'FC$ meets the plane through OEA perpendicular to the given plane. In other words, if

the axis of the first rotation lies in a given plane, the axis of the second will pass through a fixed point (not generally in the given plane). It is not difficult to see how to modify these proofs for the case in which it is the axis of the second rotation, which either passes through a fixed point or lies in a given plane.

NOTE ON A ROTATING LIQUID ELLIPSOID.

By *J. P. Johnston, M.A.*

THE following method of proving that if a liquid ellipsoid is rotating round an axis through its centre of inertia under the action of self-attraction as a rigid body, the axis of rotation must be its least principal axis, may be of interest.

We will assume that the axis of rotation must be one of the principal axes of the ellipsoid, of which there is a very simple proof due to Professor Greenhill.*

Divide the ellipsoid into a series of elliptic homœoids and take a point on the surface of the ellipsoid which lies in either of the principal planes which pass through the least principal axis. The direction of the attraction at this point of any of the shells is the internal bisector of the angle formed by joining the point to the foci of the section of the shell made by the principal plane; for by Poisson's theorem it is the internal axis of the tangent cone drawn from the point to the shell, which from symmetry is the bisector of the angle formed by the tangent lines from the point to the section of the shell. Since of the sections of any two of the shells the foci of the inner are nearer the centre than those of the outer, it is obvious that the bisector of the angle formed by the lines connecting the point with the foci of the inner is nearer the centre than the bisector of the corresponding angle for the outer. The bisector of the corresponding angle for the outer-most shell is the normal at the point. Consequently the tangential component of attraction of the whole ellipsoid at the point is along the tangent line to the meridian and *towards* the least axis. If the ellipsoid were rotating round the major axis of the section the tangential component of the acceleration at the point due to the rotation would be in the same direction, and therefore the ellipsoid could not be rotating as a rigid body.

* Besant, *Hydromechanics*, 3rd ed., p. 146.