are $\frac{1}{2}(-\sin^2 A + \sin^2 B + \sin^2 C)$, $\frac{1}{2}(\sin^2 A - \sin^2 B + \sin^2 C)$, $\frac{1}{2}(\sin^2 A + \sin^2 B - \sin^2 C)$ respectively. But we have

$$A + B + C = \pi,$$

and thence

$$\begin{aligned} & \sin^2 A + \sin^2 B - \sin^2 C, \\ = \; & \sin^2 A + \sin^2 B - \sin^2 (A+B) \\ = \; & 2 \sin A \sin B (\sin A \sin B - \cos A \cos B), \\ = \; & -2 \sin A \sin B \cos (A+B), \\ = \; & 2 \sin A \sin B \cos C, \end{aligned}$$

and we thus have

$$a, b, c = \sin B \sin C \cos A, \ \sin C \sin A \cos B, \ \sin A \sin B \cos C,$$

(or, what is the same thing, $a : b : c = \cot A : \cot B : \cot C$), and the equation of the circle is

$$yz \sin^2 A + zx \sin^2 B + xy \sin^2 C$$
$$-\tfrac{1}{2}(x \sin B \sin C \cos A + y \sin C \sin A \cos B + z \sin A \sin B \cos C)$$
$$\times (x + y + z) = 0.$$

We thus have $x : y : z = 1 : 1 : 1$ for the point O', and $x : y : z = \cot A : \cot B : \cot C$ for the point O; viz O' is the point of intersection of the lines from the angles to the mid-points of the opposite sides respectively; and O is the point of intersection of the perpendiculars from the angles on the opposite sides respectively: and the foregoing equation is consequently that of the Nine-points Circle.

ON THE NINE-POINTS CIRCLE OF A PLANE TRIANGLE.

By Professor *Cayley*.

I CONSIDER the circle which meets the sides of a triangle ABC in the points F, L; G, M; H, N respectively, where ultimately F, G, H are the feet of the perpendiculars let fall from the angles on the opposite sides, and L, M, N are the mid-points of the sides: but in the first instance they are taken to be arbitrary points. Taking the radius of the circle to be unity, the coordinates of the point F may be taken to be $\cos F$, $\sin F$, and these may be expressed rationally in terms of the tangent of the half-angle, $f = \tan \frac{1}{2}F$; and similarly for the other points, viz. we may determine the six points by the parameters f, g, h, l, m, n respectively. The sides of the triangle are the lines joining the points L, F; M, G; N, H respectively: thus the equations of the sides are

$$\begin{aligned}
&\text{for } BC \quad x(1-lf)+y(l+f)-(1+lf)=0, \text{ say } U=0,\\
&\text{,, } CA \quad x(1-mg)+y(m+g)-(1+mg)=0, \quad \text{,, } V=0,\\
&\text{,, } AB \quad x(1-nh)+y(n+h)-(1+nh)=0, \quad \text{,, } W=0.
\end{aligned}$$

We have AF a line through the intersections of BC and CA; its equation is therefore of the form $BV - CW = 0$, and to determine B, C we have $BV_0 - CW_0 = 0$, if V_0, W_0 are the values of V, W belonging to the point F, the coordinates of which are $\frac{1-f^2}{1+f^2}$, $\frac{2f}{1+f^2}$; we find

$$V_0 = -2(f-g)(f-m) \div (1+f^2);$$

$$W_0 = \;\; 2(h-f)(f-n) \div (1+f^2),$$

and then $B \div C = W_0 \div V_0$: we thus find

$$\begin{aligned}
\text{equation} \qquad &AF \text{ is } BV - CW = 0,\\
&BG \text{ ,, } C'W - A'U = 0,\\
&CH \text{ ,, } A''U - B''V = 0.
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad B : C &= -(h-f)(f-n) : (f-g)(f-m),\\
C' : A' &= -(f-g)(g-l) : (g-h)(g-n),\\
A'' : B'' &= -(g-h)(h-m) : (h-f)(h-l).
\end{aligned}$$

The condition in order that the three lines may meet in a point is $CB'A'' = CA'B''$, viz. this is

$$(f-n)(g-l)(h-m) + (f-m)(g-n)(h-l) = 0,$$

or, as this may also be written,

$$\begin{aligned}
2fgh - gh(m+n) - hf(n+l) - fg(l+m)&\\
+ mn(g+h) + nl(h+f) + lm(f+g) - 2lmn &= 0.
\end{aligned}$$

Similarly

equation AL is $\mathfrak{B}V - \mathfrak{C}W = 0$,

BM „ $\mathfrak{C}'W - \mathfrak{A}'U = 0$,

CN „ $\mathfrak{A}''U - \mathfrak{B}''V = 0$.

where $\mathfrak{B} : \mathfrak{C} = -(n-l)(h-l) : (l-m)(g-l)$,

$\mathfrak{C}' : \mathfrak{A}' = -(l-m)(f-m) : (m-n)(h-m)$,

$\mathfrak{A}'' : \mathfrak{B}'' = -(m+n)(g-n) : (n-l)\ (f-n)$,

and the condition in order that the three lines may meet in a point is $\mathfrak{B}\mathfrak{C}'\mathfrak{A}'' = \mathfrak{C}\mathfrak{A}'\mathfrak{B}''$, viz. this is the same condition as before; that is if the lines AF, BG, CH meet in a point, then also the lines AL, BM, CN will meet in a point.

In the case of the nine-points circle we have MN, NL, LM parallel to LF, MG, NH respectively: the equation of MN is

$$x(l-mn) + y(m+n) - (l+mn) = 0,$$

and this is parallel to LF, if

$$\frac{m+n}{1-mn} = \frac{l+f}{1-lf}, \text{ that is } L+F = M+N.$$

Hence for the nine-points circle we have

$$L+F = M+N,\ M+G = N+L,\ N+H = L+M,$$

or, as these equations may be written,

$$2L = G+H,\ 2M = H+F,\ 2N = F+G,$$

viz. it thus appears that the radii to the points L, M, N respectively, or say the radii L, M, N, bisect the angles made by the radii G and H, H and F, F and G respectively.

It may be added that we have

$$m+n-l+lmn = f\{1-mn+l(m+n)\},$$

$$n+l-m+lmn = g\{1-nl+m(n+l)\},$$

$$l+m-n+lmn = h\{1-lm+n(l+m)\},$$

viz. f, g, h are expressible each of them as a rational function of l, m, n.