Thus $e^{\frac{x}{y}\Sigma\Omega} S(a_0, a_1', a_2'', \ldots, a_n^{(n)})\, y^W$ is an absolute covariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$. And $S(a_0, a_1, a_2, \ldots, a_n)\, y^W$ is what this becomes when we remove all accents.

The resultant of this covariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$ and α_1', *i.e.* $a_0'x + a_1'y$, is an invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$. Now this resultant is

$$R \equiv \left\{a_0'^W - a_0'^{W-1} a_1' \Sigma\Omega + \frac{1}{1.2} a_0'^{W-2} a_1'^2 (\Sigma\Omega)^2 - \ldots \right.$$

$$\left. + (-1)^W \frac{1}{W!} a_1'^W (\Sigma\Omega)^W \right\} S(a_0, a_1', a_2'', \ldots, a_n^{(n)}).$$

And what this becomes when we remove all accents is an invariant

$$a_0^W S(a_0, a_1, a_2, \ldots, a_n)$$

of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$.

We see then that $a_0^W S$ is an integral invariant of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ given by an integral invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$, but that S itself, though an integral invariant of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ is integral only in consequence of the cancelling against one another of its fractional parts in virtue of the special equalities among the coefficients, being in fact the representative of the, as a rule, fractional invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$,

$$a_0^{-W} R.$$

ON TWISTED CUBICS AND THE CUBIC TRANSFORMATION OF ELLIPTIC FUNCTIONS.

By *A. C. Dixon, M.A.*

In the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXIII, p. 352, it is found that the modular equation for the cubic transformation of elliptic functions expresses the condition that four straight lines should touch the same twisted cubic curve. The question naturally arises, have the elliptic functions themselves any connexion with the matter? An answer to that question may be given as follows:—

Let the parameters of two points on the cubic be θ and ϕ. Then the six coordinates of the chord joining them are of the second degree in θ and ϕ separately. The condition that this chord meet a fixed straight line is therefore a doubly quadratic equation. Conversely a symmetrical doubly quadratic equation between θ and ϕ is the condition that the chord $\theta\phi$ belongs to a certain linear complex, as it is a linear equation connecting the coordinates of the line.

Now such an equation may be satisfied by putting θ and ϕ equal to the same linear fractional function of $\mathrm{sn}^2 u$ and $\mathrm{sn}^2(u+a)$, a being a constant properly chosen. In the particular case when $\theta\phi$ is to meet a given line, the equation is satisfied by θ and ϕ, or ϕ and χ, or χ and θ if $\theta\phi\chi$ is any plane through the line. We must therefore have

$$\mathrm{sn}^2(u+3a) = \mathrm{sn}^2 u,$$

and $3a$ is a period, say ω. We then have $\mathrm{sn}(u+\omega) = \mathrm{sn}\, u$, without loss of generality.

Take then the cubic as generated by the motion of the point

$$(1,\ \mathrm{sn}^2 u,\ \mathrm{sn}^4 u,\ \mathrm{sn}^6 u).$$

The plane containing this point and $u \pm \frac{1}{3}\omega$ has for its equation

$$\begin{aligned} x\,\mathrm{sn}^2 u\ \mathrm{sn}^2(u+\tfrac{1}{3}\omega)\ \mathrm{sn}^2(u-\tfrac{1}{3}\omega) - y\,\{&\mathrm{sn}^2(u+\tfrac{1}{3}\omega)\ \mathrm{sn}^2(u-\tfrac{1}{3}\omega) \\ &+ \mathrm{sn}^2 u\ \mathrm{sn}^2(u+\tfrac{1}{3}\omega) + \mathrm{sn}^2 u\ \mathrm{sn}^2(u-\tfrac{1}{3}\omega)\} \\ &+ z\,\{\mathrm{sn}^2 u + \mathrm{sn}^2(u+\tfrac{1}{3}\omega) + \mathrm{sn}^2(u-\tfrac{1}{3}\omega)\} - w = 0. \end{aligned}$$

Now we may put

$$\mathrm{sn}\, u\ \mathrm{sn}(u+\tfrac{1}{3}\omega)\ \mathrm{sn}(u-\tfrac{1}{3}\omega) = g\ \mathrm{sn}(Mu, \lambda),$$

using a cubic transformation.

It follows that

$$\mathrm{sn}\, u + \mathrm{sn}(u+\tfrac{1}{3}\omega) + \mathrm{sn}(u-\tfrac{1}{3}\omega) = h\,\mathrm{sn}(Mu, \lambda),$$

and $\quad g : h :: -\mathrm{sn}^2\tfrac{1}{3}\omega : 1 + 2\,\mathrm{cn}\tfrac{1}{3}\omega\ \mathrm{dn}\tfrac{1}{3}\omega :: 1 : -k^2\mathrm{sn}^2\tfrac{1}{3}\omega.$

We also have

$$\begin{aligned} \mathrm{sn}(u+\tfrac{1}{3}\omega)\ \mathrm{sn}(u-\tfrac{1}{3}\omega) + \mathrm{sn}\, u\ \mathrm{sn}(u+\tfrac{1}{3}\omega) + \mathrm{sn}\, u\ \mathrm{sn}(u-\tfrac{1}{3}\omega) \\ = \text{a constant} = -\mathrm{sn}^2\tfrac{1}{3}\omega. \end{aligned}$$

Thus the coefficient of z is $h^2\mathrm{sn}^2(Mu, \lambda) + 2\,\mathrm{sn}^2\frac{1}{3}\omega$, and that of $-y$ is

$$\mathrm{sn}^4\tfrac{1}{3}\omega - 2gh\ \mathrm{sn}^2(Mu, \lambda).$$

The equation becomes accordingly

$$(x - 2yk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega + zk^4 \operatorname{sn}^4 \tfrac{1}{3}\omega)\, g^2 \operatorname{sn}^2 (Mu, \lambda) - (y \operatorname{sn}^4 \tfrac{1}{3}\omega - 2z \operatorname{sn}^2 \tfrac{1}{3}\omega + w) = 0.$$

Thus the fixed line is represented by

$$x - 2yk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega + zk^4 \operatorname{sn}^4 \tfrac{1}{3}\omega = 0,$$
$$y \operatorname{sn}^4 \tfrac{1}{3}\omega - 2z \operatorname{sn}^2 \tfrac{1}{3}\omega + w = 0.$$

Again, the equation to the chord joining the points $u \pm \frac{1}{3}\omega$ are

$$x \operatorname{sn}^2 (u + \tfrac{1}{3}\omega) \operatorname{sn}^2 (u - \tfrac{1}{3}\omega) - y \{\operatorname{sn}^2 (u + \tfrac{1}{3}\omega) + \operatorname{sn}^2 (u - \tfrac{1}{3}\omega)\} + z = 0,$$
$$y \operatorname{sn}^2 (u + \tfrac{1}{3}\omega) \operatorname{sn}^2 (u - \tfrac{1}{3}\omega) - z \{\operatorname{sn}^2 (u + \tfrac{1}{3}\omega) + \operatorname{sn}^2 (u - \tfrac{1}{3}\omega)\} + w = 0.$$

The first may be written

$$x (\operatorname{sn}^2 u - \operatorname{sn}^2 \tfrac{1}{3}\omega)^2 - 2y (\operatorname{sn}^2 u \operatorname{cn}^2 \tfrac{1}{3}\omega \operatorname{dn}^2 \tfrac{1}{3}\omega + \operatorname{sn}^2 \tfrac{1}{3}\omega \operatorname{cn}^2 u \operatorname{dn}^2 u) + z (1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 \tfrac{1}{3}\omega)^2 = 0,$$

which, combined with

$$x - 2yk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega + zk^4 \operatorname{sn}^4 \tfrac{1}{3}\omega = 0,$$

gives $\quad x \operatorname{sn}^4 \frac{1}{3}\omega - 2y \operatorname{sn}^2 \frac{1}{3}\omega + z$

$$= 2 \operatorname{sn}^2 u \,[x \operatorname{sn}^2 \tfrac{1}{3}\omega + y (\operatorname{cn}^2 \tfrac{1}{3}\omega \operatorname{dn}^2 \tfrac{1}{3}\omega - \operatorname{sn}^2 \tfrac{1}{3}\omega - k^2 \operatorname{sn}^2 \tfrac{1}{3}\omega) + zk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega]$$

or

$$x (1 + k^2 \operatorname{sn}^4 \tfrac{1}{3}\omega) - 2y \operatorname{sn}^2 \tfrac{1}{3}\omega = \operatorname{sn}^2 u \,\{2zk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega - y (1 + k^2 \operatorname{sn}^4 \tfrac{1}{3}\omega)\}.$$

This gives the point at which the chord meets the fixed line, and a comparison with the equation to the plane shews that the cubic transformation of elliptic functions expresses the relation between a plane passing through a fixed line and the intersection with that line of one of the three chords of a twisted cubic that lie in the plane.

The four tangents that the line meets are given by putting the values $0, K, K', K + K'$ for u; the values of $\operatorname{sn}^2 u$ are $0, 1, \infty, 1/k^2$, and of $\operatorname{sn}^2 (Mu, \lambda)$ are $0, 1, \infty, 1/\lambda^2$.

Hence k^2 is the anharmonic ratio of the four points in which the tangent meet the line and λ^2 that of the four planes through the line which contain them, that is of the four points in which they meet the conjugate line. These two anharmonic ratios should therefore be connected by the modular equation of the cubic transformation, as was found to be the case.