ON SERIES INVOLVING INVERSE EVEN POWERS OF SUBEVEN AND SUPEREVEN NUMBERS.

By *J. W. L. Glaisher*.

Introduction, § 1.

§ 1. In a paper in Vol. XXVI. of the *Quarterly Journal** I was led to calculate the logarithms of u_2, u_6, u_{10}, ..., where

$$u_n = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \&c.$$

The value of $\log u_2$ was there deduced from a twenty-figure value of u_2, which I had calculated some years before†, but the values of $\log u_6$, $\log u_{10}$, ... were calculated without previously determining the values of u_6, u_{10}, ... In § 24 (p. 42) of the paper I remarked that, in order to calculate $\log u_4$, it would probably be found convenient first to calculate u_4 by means of Euler's semi-convergent series

$$\Sigma u_x = C + \int u_x dx - \tfrac{1}{2} u_x + \frac{B_1}{2\,!}\frac{du_x}{dx} - \frac{B_2}{4\,!}\frac{d^3 u_x}{dx^3} + \&c.$$

At the time of writing that paper I had not noticed that u_2, u_4, u_6, ... could all be calculated very readily by the following method, which was suggested by the processes employed in the preceding paper on numerical products. The same method also applies to the series

$$1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \&c.,$$

and generally to series of even inverse powers of subeven and supereven numbers, having contrary signs; the subeven numbers to mod. a being those which $\equiv -1$, mod. a, and the supereven numbers to mod. a those which $\equiv 1$, mod. a‡.

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* 'On the series $\frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} - \frac{1}{13^2} - \&c.$,' pp. 33–47.

† 'On the numerical value of a certain series.' *Proc. Lond. Math. Soc.*, Vol. VIII., pp. 200–204.

‡ *Quart. Jour. Math.*, Vol. XXVI., p. 64.

General series involving inverse squares, §§ 2, 3.

§ 2. We have

$$\log\left\{\left(\frac{1+x}{1-x}\right)\left(\frac{2+x}{2-x}\right)\dots\left(\frac{n+x}{n-x}\right)\right\} = 2S_1x + \tfrac{2}{3}S_3x^3 + \tfrac{2}{5}S_5x^5 + \&c.,$$

where, as in the preceding paper, if $r > 1$,

$$S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + \&c.;$$

and

$$S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$
$$= \gamma + \log n.$$

§ 3. Differentiating with respect to x,

$$\frac{1}{1+x} + \frac{1}{1-x} + \frac{1}{2+x} + \frac{1}{2-x} + \dots + \frac{1}{n+x} + \frac{1}{n-x}$$
$$= 2\,(S_1 + S_3x^2 + S_5x^4 + S_7x^6 + \&c.);$$

and, differentiating again,

$$\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} - \frac{1}{(2+x)^2} + \frac{1}{(3-x)^2} - \frac{1}{(3+x)^2} + \&c.$$
$$= 4\,(S_3x + 2S_5x^3 + 3S_7x^5 + \&c.).$$

The series u_2, §§ 4–8.

§ 4. Putting $x = \frac{1}{4}$, we find

$$\frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2} + \&c.$$
$$= \frac{1}{4}\left(S_3\frac{1}{4} + 2S_5\frac{1}{4^3} + 3S_7\frac{1}{4^5} + \&c.\right)$$
$$= S_3\frac{1}{4^2} + 2S_5\frac{1}{4^4} + 3S_7\frac{1}{4^6} + 4S_9\frac{1}{4^8} + \&c.$$

§ 5. This series is even more convenient for calculation than that which was used in Vol. VIII. of *Proc. Lond. Math. Soc.*, viz.

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \&c.$$
$$= \frac{\pi}{2}\left(\frac{1}{2} + \frac{1}{3}\frac{S_2}{2^2} + \frac{1}{5}\frac{S_4}{2^4} + \frac{1}{7}\frac{S_6}{2^6} + \&c.\right),$$

where $$s_r = 1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \&c.$$

§ 6. In order to express the series in a form better adapted to actual calculation, the S's should be replaced by S''s, where $S'_r = S_r - 1$. This is effected by simply omitting the first two terms in § 3, the formula being

$$\frac{1}{(2-x)^2} - \frac{1}{(2+x)^2} + \frac{1}{(3-x)^2} - \frac{1}{(3+x)^2} + \&c.$$
$$= 4\,(S_3'x + 2S_5'x^3 + 3S_7'x^5 + \&c.).$$

§ 7. Putting $x = \frac{1}{4}$, we thus find

$$\frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{13^2} - \&c. = S_3'\frac{1}{4^2} + 2S_5'\frac{1}{4^4} + 2\,S_7'\frac{1}{4^6} + \&c.$$

§ 8. Denoting, as in the paper in the *Quarterly Journal*, the series

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \&c.$$

by u_n, the results of §§ 4 and 7 may be written

$$1 - u_2 = S_3\frac{1}{4^2} + 2S_5\frac{1}{4^4} + 3S_7\frac{1}{4^6} + \&c.$$

$$1 - \frac{1}{9} + \frac{1}{25} - u_2 = S_3'\frac{1}{4^2} + 2S_5'\frac{1}{4^4} + 3S_7'\frac{1}{4^6} + \&c.$$

The series g_2, §§ 9—10.

§ 9. Putting $x = \frac{1}{3}$ in the formula of § 3, we have

$$\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c. = 4\left(S_3\frac{1}{3^3} + 2S_5\frac{1}{3^5} + 3S_7\frac{1}{3^7} + \&c.\right).$$

§ 10. In a paper* in Vol. XXVI. of the *Quarterly Journal*, the series

$$1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \&c.$$

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* "On the series $\frac{1}{2} + \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \&c.$," pp. 48—65. In this paper the series g_n is considered, but only for uneven values of n.

was denoted by g_n. Using this notation the formula of the preceding section may be written

$$1 - g_2 = 4\left(S_3 \frac{1}{3^3} + 2S_5 \frac{1}{3^5} + 3S_7 \frac{1}{3^7} + \&c.\right).$$

Corresponding to the formula of § 7, we have also

$$1 - \frac{1}{4} + \frac{1}{16} - g_2 = 4\left(S_3' \frac{1}{3^3} + 2S_5' \frac{1}{3^5} + 3S_7' \frac{1}{3^7} + \&c.\right).$$

Values of u_4 and g_4, §§ 11, 12.

§ 11. Differentiating twice the formula in § 3, we have

$$\frac{3!}{(1-x)^4} - \frac{3!}{(1+x)^4} + \frac{3!}{(2-x)^4} - \frac{3!}{(2+x)^4} + \&c.$$
$$= 2\,\{2.3.4S_5 x + 4.5.6S_7 x^3 + 6.7.8S_9 x^5 + \&c.\},$$

so that

$$\frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} - \frac{1}{(2+x)^4} + \&c.$$
$$= \tfrac{1}{3}\,\{2.3.4S_5 x + 4.5.6S_7 x^3 + 6.7.8S_9 x^5 + \&c.\}.$$

§ 12 Putting $x = \frac{1}{4}$ and $x = \frac{1}{3}$, this formula gives

$$\frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} - \frac{1}{9^4} + \&c. = \tfrac{1}{3}\left\{2.3.4S_5 \frac{1}{4^5} + 4.5.6S_7 \frac{1}{4^7} + \&c.\right\},$$

and

$$\frac{1}{2^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{7^4} + \&c. = 2.3.4S_5 \frac{1}{3^6} + 4.5.6S_7 \frac{1}{3^8} + \&c.,$$

and we have of course the corresponding formulæ involving S''s for

$$1 - \frac{1}{3^4} + \frac{1}{5^4} - u_4 \text{ and } 1 - \frac{1}{2^4} + \frac{1}{4^4} - g_4.$$

Values of u_6 and g_6, §§ 13, 14.

§ 13. Similarly we find

$$\frac{5!}{(1-x)^6} - \frac{5!}{(1+x)^6} + \frac{5!}{(2-x)^6} - \frac{5!}{(2+x)^6} + \&c.$$
$$= 2\,\{2.3.4.5.6S_7 x + 4.5.6.7.8S_9 x^3 + 6.7.8.9.10S_{11} x^5 + \&c.\},$$

so that

$$\frac{1}{3^6} - \frac{1}{5^6} + \frac{1}{7^6} - \frac{1}{9^6} + \&c.$$

$$= \tfrac{1}{15}\left\{2.3.4.5.6S_7\frac{1}{4^7} + 4.5.6.7.8S_9\frac{1}{4^9} + \&c.\right\},$$

and

$$\frac{1}{2^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{7^6} + \&c.$$

$$= \tfrac{1}{20}\left\{2.3.4.5.6.S_7\frac{1}{3^8} + 4.5.6.7.8S_9\frac{1}{3^{10}} + \&c.\right\}.$$

It is unnecessary to write down the corresponding formulæ for the higher powers, as the general law is obvious.

Series involving subeven and supereven numbers, § 14.

§ 14. It is evident that, by putting $x = \frac{1}{a}$ in the general formulæ, we obtain expressions for the series of which the terms are the inverse even powers of subeven and supereven numbers to any modulus, taken with different signs.

Thus,

$$\frac{1}{(a-1)^2} - \frac{1}{(a+1)^2} + \frac{1}{(2a-1)^2} - \frac{1}{(2a+1)^2} + \&c.$$

$$= 4\left(S_3\frac{1}{a^3} + 2S_5\frac{1}{a^5} + 3S_7\frac{1}{a^7} + \&c.\right),$$

$$\frac{1}{(a-1)^4} - \frac{1}{(a+1)^4} + \frac{1}{(2a-1)^4} - \frac{1}{(2a+1)^4} + \&c.$$

$$= \tfrac{1}{3}\left(2.3.4S_5\frac{1}{a^5} + 4.5.6S_7\frac{1}{a^7} + \&c.\right),$$

$$\frac{1}{(a-1)^6} - \frac{1}{(a+1)^6} + \frac{1}{(2a-1)^6} - \frac{1}{(2a+1)^6} + \&c.$$

$$= \tfrac{1}{60}\left(2.3.4.5.6S_7\frac{1}{a^7} + 4.5\,6.7.8S_9\frac{1}{a^9} + \&c.\right),$$

and so on.

Series involving uneven multiples of x only, §§ 15—18.

§ 15. Proceeding as in § 2 and starting with

$$\log\left\{\left(\frac{1+x}{1-x}\right)\left(\frac{2+x}{2-x}\right)\cdots\left(\frac{2n-1+x}{2n-1-x}\right)\right\}$$

$$= 2U_1x + \tfrac{2}{3}U_3x^3 + \tfrac{2}{5}U_5x^5 + \&\text{c.},$$

where, as in the previous paper (§ 36),

$$U_r = 1 + \frac{1}{3^r} + \frac{1}{5^r} + \frac{1}{7^r} + \&\text{c.},$$

$$\text{and } U_1 = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}$$

$$= \tfrac{1}{2}\gamma + \log 2 + \tfrac{1}{2}\log n,$$

we find, by differentiating twice,

$$\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(3-x)^2} - \frac{1}{(3+x)^2} + \&\text{c.}$$

$$= 4(U_3x + 2U_5x^3 + 3U_7x^5 + \&\text{c.}).$$

§ 16. Putting $x = \frac{1}{2}$, we find

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \&\text{c.}$$

$$= U_3\frac{1}{2} + 2U_5\frac{1}{2^3} + 3U_7\frac{1}{2^5} + \&\text{c.}$$

and also, writing

$$U_3' = U_3 - 1,\ \ U_5' = U_5 - 1,\ \&\text{c.},$$

$$\frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \&\text{c.} = U_3'\frac{1}{2} + 2U_5\frac{1}{2^3} + 3U_7'\frac{1}{2^5} + \&\text{c.},$$

the left-hand members of these equations being

$$u_2 \text{ and } u_2 - 1 - \frac{1}{3^2}$$

respectively.

These series do not converge so rapidly as those given in § 8, which proceed by powers of $\frac{1}{4}$ intead of $\frac{1}{2}$.

§ 17. We find also, by differentiation, as in §§ 11 and 13,

$$\frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(3-x)^4} - \frac{1}{(3+x)^4} + \&c.$$
$$= \tfrac{1}{3}\{2.3.4\,U_5x + 4.5.6\,U_7x^3 + \&c.\};$$

$$\frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(3-x)^6} - \frac{1}{(3+x)^6} + \&c.$$
$$= \tfrac{1}{15}\{2.3.4.5.6\,U_7x + 4.5.6.7.8\,U_9x^3 + \&c.\},$$

and so on; the formulæ being exactly similar to those in §§ 11 and 13, but with U's in place of S's.

§ 18. By putting $x = \frac{1}{2}$, we obtain expressions for

$$g_4,\ g_4 - 1 + \frac{1}{3^4},$$

$$g_6,\ g_6 - 1 + \frac{1}{3^6},\ \&c.,$$

in series proceeding by powers of $\frac{1}{2}$.

Remarks on the formulæ, §§ 19–20.

§ 19. The formula of § 8 affords a striking example of the great simplification in the calculation of a numerical quantity, which may be effected by an algebraical transformation of the most elementary character. Four terms of the S'-series in that section suffice to give the value of u_2 to nine places of decimals, and the calculation does not require five minutes' work. But the nine-place value given in Vol. VI. of the *Messenger* (p. 76) was only obtained as the result of a laborious calculation; and if we calculate u_2 directly from the series itself, it is necessary to include terms up to $\frac{1}{315^2}$ in order to obtain five places (p. 74). The series in § 8 is also preferable to that used in Vol. VIII. of the *Proc. Lond. Math. Soc.* as it converges much more rapidly, and does not require the final multiplication by $\frac{1}{2}\pi$.

§ 20. The extreme simplicity of the formulæ of transformation is noticeable. In the preceding sections they have been deduced from the logarithmic products of §§ 2 and 15, because it was in this way that I was led to them, and it seemed interesting to connect them with the formulæ of the preceding paper. The truth of each formulæ of transformation is, however, evident at sight; *e.g.* the equation

$$\frac{1}{(1-x)^4} - \frac{1}{(1+x)} + \frac{1}{(2-x)^4} - \frac{1}{(2+x)^4} - \&c.$$

$$= \tfrac{1}{3}\,(2.3.4S_5x + 4.5.6S_7x^3 + \&c.),$$

or

$$\frac{1}{(2-x)^4} - \frac{1}{(2+x)^4} + \frac{1}{(3-x)^4} - \frac{1}{(3+x)^4} + \&c.$$

$$= \tfrac{1}{3}\,(2.3.4S_5'x + 4.5.6S_7'x^3 + \&c.)$$

is at once seen to be true, by expanding the left-hand side in ascending powers of x by the Binomial Theorem.

Relation connecting u_2, u_4, u_6, &c., § 21.

§ 21. In the *Nouvelles Annales* (Ser. III. Vol. II., p. 429), it was shown that

$$\left\{\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right\}^2 = 8\,\frac{9}{8}\,\frac{24}{25}\,\frac{49}{48}\,\frac{80}{81}\,\frac{121}{120}\ldots,$$

now $$\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \frac{\pi}{\sin\frac{1}{4}\pi} = \pi\sqrt{2},$$

so that the left-hand side

$$= \frac{\Gamma^4(\frac{1}{4})}{2\pi^2}.$$

The equation may therefore be written

$$\frac{\Gamma^4(\frac{1}{4})}{16\pi^3} = \frac{3^2}{3^2-1}\,\frac{5^2-1}{5^2}\,\frac{7^2}{7^2-1}\,\frac{9^2-1}{9^2}\ldots.$$

Now, if K^0 denote the complete elliptic integral of the first kind to modulus $\dfrac{1}{\sqrt{2}}$,

$$K^0 = \frac{\Gamma^2(\frac{1}{4})}{4\pi^{\frac{1}{2}}},$$

whence $$\frac{(K^0)^2}{\pi} = \frac{3^2}{3^2-1}\,\frac{5^2-1}{5^2}\,\frac{7^2}{7^2-1}\,\frac{9^2-1}{9^2}\ldots,$$

and by taking the logarithm, we have

$$\log \frac{(K^0)^2}{\pi} = -\log\left(1-\frac{1}{3^2}\right) + \log\left(1-\frac{1}{5^2}\right) - \log\left(1-\frac{1}{7^2}\right) + \&c.$$

$$= \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2} + \&c.$$

$$+ \tfrac{1}{2}\left(\frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} - \frac{1}{9^4} + \&c.\right)$$

$$+ \tfrac{1}{3}\left(\frac{1}{3^6} - \frac{1}{5^6} + \frac{1}{7^6} - \frac{1}{9^6} + \&c.\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 1 - u_2 + \tfrac{1}{2}(1-u_4) + \tfrac{1}{3}(1-u_6) + \&c.,$$

so that, denoting $1 - u_n$ by u_n',

$$u_2' + \tfrac{1}{2}u_4' + \tfrac{1}{3}u_6' + \&c. = 2\log K^0 - \log \pi.$$

This formula would afford a very useful verification of the values of u_2, u_4, u_6, ... when calculated; or it could be applied to the calculation of u_4 say, when u_6, u_8... had been calculated.

§ 22. The quantity K^0 is equal to $\dfrac{\omega}{\sqrt{2}}$, where w is the quantity so denoted by Gauss* (*i.e.*, the length of a lemniscate whose diameter is unity).

Thus the right-hand member of the equation

$$= 2\log\omega - \log 2 - \log\pi.$$

Gauss calculated $\log\omega$ to twenty-five places of decimals, his result being†

$$\log\omega = 0{\cdot}96395\ 93356\ 31536\ 86352\ 36577.$$

Taking this value of $\log\omega$, the value of the series

$$u_2' + \tfrac{1}{2}u_4' + \tfrac{1}{3}u_6' + \tfrac{1}{4}u_8' + \&c.$$

is $$0{\cdot}09004\ 16048\ 53728\ 24348\ 66558.$$

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* *Werke*, Vol. III., p. 413. † *Id.*, p. 414.