O wzorach

służących do obliczenia liczby liczb pierwszych

nie przekraczających danej granicy.

Przez

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Od lat kilku zupełnie zależałem pole gramatyki, a ugrzązłem, raczej pogrążyłem się w matematyce, nie jako fachowy, ale jak dyletant, samouczek, bez prawideł, wskazówek i znajomości powszechnie przyjętych formuł. Napisałem kwadraty i sześciany liczb od 1 do 1000 z ich różnicami. Potęgowałem liczby małe: 2, 2^2, 2^3... 2^{300}; 3, 3^2, 3^3... 3^{200}; 5, 5^2, 5^3... 5^{150}; 7, 7^2, 7^3 . . . 7^{85}. Przy tej pracy poznałem stosunki i prawidła, o których później się dowiedziałem, że nazywane są dwumianem Newton'a. Zastanawiając się nad stosunkami i prawami wykładników, wpadłem na trop, który mię doprowadził do zbadania wielokształtu podzielności liczb złożonych. Przytem zbadałem wszystkie liczby w zakresie numeracyi od 0 do 1000; potem do 10000. Za wskazówką i poradą Dra Webera zapisałem sobie dzieło Wertheima „Die Theorie der Zahlen", ale jego wykład i znakowanie — dla mnie, jako laika, były orzechami zanadto twardemi na moje zęby. Zainteresowała mię jednak funkcya $\varphi(m)$, jako mająca związek z moją analizą liczb w celu znalezienia prawideł, zapomocą których mógłbym poznać, która liczba jest pierwszą,

a jakie są cechy liczb złożonych. Przy pomocy Dra Hossfelda w Eisenach, poznałem zasady funkcyi $\varphi(m)$. Opracowując tę funkcyę w zakresie do 100 000, potem do 1,000 000, wykryłem okresy symetrycznych luk między liczbami względnie pierwszemi, po usunięciu każdej z porządku z liczb pierwszych ze wszystkiemi przez nią podzielnemi liczbami i prawidło ogólne, że

$$\varphi(p_1 \cdot p_2 \cdot p_3 \cdots p_n, n) = (p_1 - 1)(p_2 - 1)(p_3 - 1) \ldots (p_n - 1)$$

oraz że $\varphi(g(p_1 \cdot p_2 \cdot p_3 \cdots p_n, n)) = g(p_1 - 1)(p_2 - 1)(p_3 - 1) \ldots (p_n - 1)$.

Wykryta zaś symetrya luk w danym okresie, t. j. że luka 1-sza = luce ostatniej, 2-ga = przedostatniej, 3-cia z początku = 3-ciej od końca i t. d. aż do połowy, czyli środka okresu, to jest do $\frac{p_1 \cdot p_2 \cdot p_3 \cdots p_n}{2}$ wskazała możność obliczenia na poczekaniu i zakresu numeracyi $0, 1, 2, 3, \ldots m = p_1 \cdot p_2 \cdot p_3 \ldots p_n + r$,

oraz $m = g \cdot p_1 \cdot p_2 \cdot p_3 \cdots p_n \pm r$,

gdyż $\varphi(p_1 \cdot p_2 \cdot p_3 \cdots p_n \pm r, n) = (p_1 - 1)(p_2 - 1)(p_3 - 1) \ldots (p_n - 1) \begin{cases} +\varphi(r, n) \\ -\varphi(r-1, n), \end{cases}$

oraz $\varphi(g.(p_1 \cdot p_2 \cdot p_3 \cdots p_n \pm r, n)) = (g(p_1 - 1)(p_2 - 1)(p_3 - 1) \ldots (p_n - 1) \begin{cases} +\varphi(r, n) \\ -\varphi(r-1, n). \end{cases}$

Zakomunikowałem te spostrzeżenia przez p. Webera Drowi Hossfeldowi, który przyznał im wielką wagę; ale potem zawiadomił mię, że już w roku 1872 Dr. Meissel te same prawa odkrył i ogłosił, tylko jeszcze nie wiedział o wewnętrznej symetryi okresów i dla tego wzór jego posiada tylko $+ r$, ale nie $- r$. Wydrukował o tem za mojem przyzwoleniem króciutki artykulik o dopełnieniu przeze mnie wzoru Meissela. Artykulik wyszedł niefortunnie, bo przed wydrukowaniem nie przysłał mi go do przejrzenia. Wsadził doń bez potrzeby własnego pomysłu p_x, bez potrzeby i sensu. Kiedym go później przekonał, iż wszystkie trudności i pomyłki wyrażenie $-\varphi(r-1, n)$ usuwa, przysłał mi do przejrzenia napisany do druku drugi artykulik o tem samem; ale jego objaśnienie $-\varphi(r-1, n), -1$, nie trafiło do mojego przekonania; moje zaś objaśnienie on uważał za naciągnięte. W ten sposób artykulik jego nie wyszedł z druku.

Dla większego jeszcze uproszczenia obliczeń funkcyi $\varphi(m)$, ułożyłem analityczną tablicę okresu p_6, t. j. $\varphi(2.\ 3.\ 5.\ 7.\ 11.\ 13, 6) = \varphi(30\,030, 6)$, która wielkie daje ułatwienie przy obliczaniu większych zakresów numeracyi.

Potem napisałem teoryę łuk, których Dr. Hossfeld nie podjął się sprawdzić.

Nareszcie usunąwszy wszystkie liczby podzielne przez 2, 3 i 5, jako łatwe do poznania, zanalizowałem zakres numeracyi 0, 1, 2, 3, ... 150060, czyli liczby wszystkie pierwsze w tym zakresie, oraz podzielne przez 7, 11, 13, 17,..., t. j. przez p_4, p_5, p_6, p_7, . . . Spisałem na ogół liczb 40008, oznaczając je właściwemi czynnikami n. p. $49=7^2$, $77 = 7.11$; $91=7.13$, $1001 = 7.\ 11.\ 13$ i t. d.

Później z tego kajetu wypisałem osobno same tylko liczby pierwsze, to jest

$$p_1=2,\ p_2=3,\ p_3=5,\ p_4=7\ \ldots\ p_{13852}=150053.$$

Gustaw Wertheim w dziele „Elemente der Zahlentheorie“ (Leipzig 1887) rozwija i przykładem objaśnia następujący wzór Meissel'a do obliczenia w danym zakresie numeracyi liczb pierwszych str. 24.

$$\psi(n) = \varphi(n, m)+m(\mu+1)+\frac{\mu(\mu-1)}{2}-1-\sum_{s=1}^{s=\mu}\psi\left(\frac{n}{p_{m+s}}\right)$$

$\psi(n)$ oznacza tutaj, ile się zawiera liczb bezwzględnie pierwszych w zakresie numeracyi od 0, 1, 2, 3, . . . n.

m oznacza, ile liczb pierwszych znajduje się w sześciennym pierwiastku zakresu n, czyli $m = \psi(\sqrt[3]{n})$.

μ oznacza, ile liczb pierwszych znajduje się w pierwiastku kwadratowym $\sqrt{n}$, po odjęciu liczby tychże liczb, będących w pierwiastku sześciennym, czyli $\mu = \psi\sqrt{n} - m$.

$$m+\mu = \psi(\sqrt{n}).$$

Wzór ten, dobry przy obliczaniu niewielkich zakresów numeracyi, kiedy n nie przewyższa setek, tysięcy; znośny jeszcze i przy obliczaniu dziesiątek tysięcy; w wielkich zaś zakresach numeracyi, wymaga wiele miejsca, czasu i pracy. Można się o tem przekonać, obliczając choćby tylko

$$n = 100000;\ \sqrt[3]{n} = 46;\ \psi(46) = 14;\ \sqrt{n} = 316;\ \psi(316) = 65;$$

ponieważ zaś $m = \psi(\sqrt[3]{n}) = \psi(46) = 14$,

przeto $\mu = [\psi(\sqrt{n}) = \psi(316) = 65] - 14 = 51$; $m + \mu = \psi(\sqrt{n}) = 65$ $\varphi(n, m) = \varphi(100\,000, 14) = 14204$. Tego trzeba dowieść rachunkiem następującym.

Liczby te 14 są następujące: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$, $p_7 = 17$, $p_8 = 19$, $p_9 = 23$, $p_{10} = 29$, $p_{11} = 31$, $p_{12} = 37$, $p_{13} = 41$, $p_{14} = 43$;

$$\varphi(100\,000, 14) = \varphi(100\,000, 13) - \left[\varphi\left(\frac{100\,000}{43}, 13\right) = \varphi(2325, 13)\right] =$$
$$= 14\,540 - 336 = \mathbf{14\,204}$$

$$\varphi(100\,000, 13) = \varphi(100\,000, 12) - \left[\varphi\left(\frac{100\,000}{41}, 12\right) = \varphi(2439, 12)\right] =$$
$$= 14\,899 - 359 = 14\,540$$

$$\varphi(100\,000, 12) = \varphi(100\,000, 11) - \left[\varphi\left(\frac{100\,000}{37}, 11\right) = \varphi(2707, 11)\right] =$$
$$= 15\,305 - 406 = 14\,899$$

$$\varphi(100\,000, 11) = \varphi(100\,000, 10) - \left[\varphi\left(\frac{100\,000}{31}, 10\right) = \varphi(3225, 10)\right] =$$
$$= 15\,805 - 500 = 15\,305$$

$$\varphi(100\,000, 10) = \varphi(100\,000, 9) - \left[\varphi\left(\frac{100\,000}{29}, 9\right) = \varphi(3448, 9)\right] =$$
$$= 16\,361 - 556 = 15\,805$$

$$\varphi(100\,000, 9) = \varphi(100\,000, 8) - \left[\varphi\left(\frac{100\,000}{23}, 8\right) = \varphi(4347, 8)\right] =$$
$$= 17\,103 - 742 = 16\,361$$

$$\varphi(100\,000, 8) = \varphi(100\,000, 7) - \left[\varphi\left(\frac{100\,000}{19}, 7\right) = \varphi(5263, 7)\right] =$$
$$= 18\,053 - 950 = 17\,103$$

$$\varphi(100\,000, 7) = \varphi(100\,000, 6) - \left[\varphi\left(\frac{100\,000}{17}, 6\right) = \varphi(5882, 6)\right] =$$
$$= 19\,181 - 1128 = 18\,053$$

$$\varphi(100\,000, 6) = \varphi(100\,000, 5) - \left[\varphi\left(\frac{100\,000}{13}, 5\right) = \varphi(7692, 5)\right] =$$
$$= 20\,779 - 1598 = 19\,181$$

$$\varphi(100\,000, 5) = \varphi(100\,000, 4) - \left[\varphi\left(\frac{100\,000}{11}, 4\right) = \varphi(9090, 4)\right] =$$
$$= 22\,857 - 2078 = 20\,779$$

$$\varphi(100000, 4) = \varphi(100000, 3) - \left[\varphi\left(\frac{100000}{7}, 3\right) = \varphi(14285, 3)\right] =$$
$$= 26666 - 3809 = 22857$$

$$\varphi(100000, 3) = \varphi(100000, 2) - \left[\varphi\left(\frac{100000}{5}, 2\right) = \varphi(20000, 2)\right] =$$
$$= 33333 - 6667 = 26666$$

$$\varphi(100000, 2) = \varphi(100000, 1) - \left[\varphi\left(\frac{100000}{3}, 1\right) = \varphi(33333, 1)\right] =$$
$$= 50000 - 16667 = 33333.$$

$$\varphi(20000, 2) = \varphi(20000, 1) - \varphi(6666, 1) = 10000 - 3333 = \mathbf{6667}$$

$$\varphi(14285, 3) = \varphi(14285, 2) - \varphi(2857, 2) = 4762 - 953 = \mathbf{3809}$$
$$\varphi(14285, 2) = \varphi(14285, 1) - \varphi(4761, 1) = 7143 - 2381 = 4762$$
$$\varphi(2857, 2) = \varphi(2857, 1) - \varphi(952, 1) = 1429 - 476 = \mathbf{953}$$

$$\varphi(9090, 4) = \varphi(9090, 3) - \varphi(1298, 3) = 2424 - 346 = \mathbf{2078}$$
$$\varphi(9090, 3) = \varphi(9090, 2) - \varphi(1818, 2) = 3030 - 606 = 2424$$
$$\varphi(9090, 2) = \varphi(9090, 1) - \varphi(3030, 1) = 4545 - 1515 = 3030$$
$$\varphi(1818, 2) = \varphi(1818, 1) - \varphi(606, 1) = 909 - 303 = \mathbf{606}$$

$$\varphi(1298, 3) = \varphi(1298, 2) - \varphi(259, 2) = 433 - 87 = \mathbf{346}$$
$$\varphi(1298, 2) = \varphi(1298, 1) - \varphi(432, 1) = 649 - 216 = 433$$
$$\varphi(259, 2) = \varphi(259, 1) - \varphi(86, 1) = 130 - 43 = \mathbf{87}$$

$$\varphi(7692, 5) = \varphi(7692, 4) - \varphi(699, 4) = 1758 - 160 = \mathbf{1598}$$
$$\varphi(7692, 4) = \varphi(7692, 3) - \varphi(1098, 3) = 2051 - 293 = 1758$$
$$\varphi(7692, 3) = \varphi(7692, 2) - \varphi(1538, 2) = 2564 - 513 = 2051$$
$$\varphi(7692, 2) = \varphi(7692, 1) - \varphi(2564, 1) = 3846 - 1282 = 2564$$
$$\varphi(1538, 2) = \varphi(1538, 1) - \varphi(512, 1) = 769 - 256 = \mathbf{513}$$

$$\varphi(1098, 3) = \varphi(1098, 2) - \varphi(219, 2) = 366 - 73 = \mathbf{293}$$
$$\varphi(1098, 2) = \varphi(1098, 1) - \varphi(366, 1) = 549 - 183 = 366$$
$$\varphi(219, 2) = \varphi(219, 1) - \varphi(73, 1) = 110 - 37 = \mathbf{73}$$

$$\varphi(699, 4) = \varphi(699, 3) - \varphi(99, 3) = 186 - 26 = \mathbf{160}$$
$$\varphi(699, 3) = \varphi(699, 2) - \varphi(139, 2) = 233 - 47 = 186$$
$$\varphi(699, 2) = \varphi(699, 1) - \varphi(233, 1) = 350 - 117 = 233$$
$$\varphi(139, 2) = \varphi(139, 1) - \varphi(46, 1) = 70 - 23 = \mathbf{47}$$

$$\varphi(99, 3) = \varphi(99, 2) - \varphi(19, 2) = 33 - 7 = \mathbf{26}$$
$$\varphi(99, 2) = \varphi(99, 1) - \varphi(33, 1) = 50 - 17 = 33$$
$$\varphi(19, 2) = \varphi(19, 1) - \varphi(6, 1) = 10 - 3 = \mathbf{7}$$

$\varphi(5882, 6) = \varphi(5882, 5) - \varphi(452, 5) = 1222 - 94 = \mathbf{1128}$
$\varphi(5882, 5) = \varphi(5882, 4) - \varphi(534, 4) = 1345 - 123 = 1222$
$\varphi(5882, 4) = \varphi(5882, 3) - \varphi(840, 3) = 1569 - 224 = 1345$
$\varphi(5882, 3) = \varphi(5882, 2) - \varphi(1176, 2) = 1961 - 392 = 1569$
$\varphi(5882, 2) = \varphi(5882, 1) - \varphi(1960, 1) = 2941 - 980 = 1961$
$\varphi(1176, 2) = \varphi(1176, 1) - \varphi(391,1) = 588 - 196 = \mathbf{392}$

$\varphi(840, 3) = \varphi(840, 2) - \varphi(168, 2) = 280 - 56 = \mathbf{224}$
$\varphi(840, 2) = \varphi(840, 1) - \varphi(280, 1) = 420 - 140 = 280$
$\varphi(168, 2) = \varphi(168, 1) - \varphi(56,1) = 84 - 28 = \mathbf{56}$

$\varphi(534, 4) = \varphi(534, 3) - \varphi(76, 3) = 143 - 20 = \mathbf{123}$
$\varphi(534, 3) = \varphi(534, 2) - \varphi(106, 2) = 178 - 35 = 143$
$\varphi(534, 2) = \varphi(534, 1) - \varphi(178, 1) = 267 - 89 = 178$
$\varphi(106, 2) = \varphi(106, 1) - \varphi(35,1) = 53 - 18 = \mathbf{35}$

$\varphi(76, 3) = \varphi(76, 2) - \varphi(15, 2) = 25 - 5 = \mathbf{20}$
$\varphi(76, 2) = \varphi(76, 1) - \varphi(25, 1) = 38 - 13 = 25$
$\varphi(15, 2) = \varphi(15, 1) - \varphi(5,1) = 8 - 3 = \mathbf{5}$

$\varphi(452, 5) = \varphi(452, 4) - \varphi(41, 4) = 104 - 10 = \mathbf{94}$
$\varphi(452, 4) = \varphi(452, 3) - \varphi(64, 3) = 121 - 17 = 104$
$\varphi(452, 3) = \varphi(452, 2) - \varphi(90, 2) = 151 - 30 = 121$
$\varphi(452, 2) = \varphi(452, 1) - \varphi(150, 1) = 226 - 75 = 151$
$\varphi(90, 2) = \varphi(90, 1) - \varphi(30,1) = 45 - 15 = \mathbf{30}$

$\varphi(64, 3) = \varphi(64, 2) - \varphi(12, 2) = 21 - 4 = \mathbf{17}$
$\varphi(64, 2) = \varphi(64, 1) - \varphi(12, 1) = 32 - 11 = 21$
$\varphi(12, 2) = \varphi(12, 1) - \varphi(4,1) = 6 - 2 = \mathbf{4}$

$\varphi(41, 4) = \varphi(41, 3) - \varphi(5, 3) = 11 - 1 = \mathbf{10}$
$\varphi(41, 3) = \varphi(41, 2) - \varphi(8, 2) = 14 - 3 = 11$
$\varphi(41, 2) = \varphi(41, 1) - \varphi(13, 1) = 21 - 7 = 14$

$\varphi(5263, 7) = \varphi(5263, 6) - \varphi(309, 6) = 1009 - 59 = \mathbf{950}$
$\varphi(5263, 6) = \varphi(5263, 5) - \varphi(404, 5) = 1094 - 85 = 1009$
$\varphi(5263, 5) = \varphi(5263, 4) - \varphi(478, 4) = 1203 - 109 = 1094$
$\varphi(5263, 4) = \varphi(5263, 3) - \varphi(751, 3) = 1404 - 201 = 1203$
$\varphi(5263, 3) = \varphi(5263, 2) - \varphi(1052, 2) = 1755 - 351 = 1404$
$\varphi(5263, 2) = \varphi(5263, 1) - \varphi(1754, 1) = 2632 - 877 = 1755$
$\varphi(1052, 2) = \varphi(1052, 1) - \varphi(350,1) = 526 - 175 = \mathbf{351}$

$\varphi(751, 3) = \varphi(751, 2) - \varphi(150, 2) = 251 - 50 = \mathbf{201}$
$\varphi(751, 2) = \varphi(751, 1) - \varphi(250, 1) = 376 - 125 = 251$
$\varphi(150, 2) = \varphi(150, 1) - \varphi(50,1) = 75 - 25 = \mathbf{50}$

$\varphi(478, 4) = \varphi(478, 3) - \varphi(68, 3) = 127 - 18 = \mathbf{109}$
$\varphi(478, 3) = \varphi(478, 2) - \varphi(95, 2) = 159 - 32 = 127$
$\varphi(478, 2) = \varphi(478, 1) - \varphi(159, 1) = 239 - 80 = 159$
$\varphi(95, 2) = \varphi(95, 1) - \varphi(31,1) = 48 - 16 = \mathbf{32}$

$\varphi(68, 3) = \varphi(68, 2) - \varphi(13,2) = 23 - 5 = \mathbf{18}$
$\varphi(68, 2) = \varphi(68, 1) - \varphi(22,1) = 34 - 11 = 23$
$\varphi(13, 2) = \varphi(13,1) - \varphi(4, 1) = 7 - 2 = \mathbf{5}$

$\varphi(404, 5) = \varphi(404, 4) - \varphi(36, 4) = 93 - 8 = \mathbf{85}$
$\varphi(404, 4) = \varphi(404, 3) - \varphi(57, 3) = 108 - 15 = 93$
$\varphi(404, 3) = \varphi(404, 2) - \varphi(80, 2) = 135 - 27 = 108$
$\varphi(404, 2) = \varphi(404, 1) - \varphi(134, 1) = 202 - 67 = 135$
$\varphi(80, 2) = \varphi(80, 1) - \varphi(26,1) = 40 - 13 = \mathbf{27}$

$\varphi(57, 3) = \varphi(57, 2) - \varphi(11,2) = 19 - 4 = \mathbf{15}$
$\varphi(57, 2) = \varphi(57, 1) - \varphi(19,1) = 29 - 10 = 19$
$\varphi(11, 2) = \varphi(11,1) - \varphi(3,1) = 6 - 2 = \mathbf{4}$

$\varphi(36, 4) = \varphi(36, 3) - \varphi(5,3) = 9 - 1 = \mathbf{8}$
$\varphi(36, 3) = \varphi(36, 2) - \varphi(7,2) = 12 - 3 = 9$
$\varphi(36, 2) = \varphi(36, 1) - \varphi(12,1) = 18 - 6 = 12$
$\varphi(7, 2) = \varphi(7,1) - \varphi(2,1) = 4 - 1 = \mathbf{3}$

$\varphi(5, 3) = \varphi(5,2) - \varphi(1,2) = 2 - 1 = \mathbf{1}$
$\varphi(5, 2) = \varphi(5,1) - \varphi(1,1) = 3 - 1 = 2$

$\varphi(309, 6) = \varphi(309, 5) - \varphi(23, 5) = 64 - 5 = \mathbf{59}$
$\varphi(309, 5) = \varphi(309, 4) - \varphi(28, 4) = 70 - 6 = 64$
$\varphi(309, 4) = \varphi(309, 3) - \varphi(44, 3) = 82 - 12 = 70$
$\varphi(309, 3) = \varphi(309, 2) - \varphi(61, 2) = 103 - 21 = 82$
$\varphi(309, 2) = \varphi(309, 1) - \varphi(103, 1) = 155 - 52 = 103$
$\varphi(61, 2) = \varphi(61, 1) - \varphi(20,1) = 31 - 10 = \mathbf{21}$

$\varphi(44, 3) = \varphi(44, 2) - \varphi(8, 2) = 15 - 3 = \mathbf{12}$
$\varphi(44, 2) = \varphi(44, 1) - \varphi(14, 1) = 22 - 7 = 15$
$\varphi(8, 2) = \varphi(8, 1) - \varphi(2, 1) = 4 - 1 = \mathbf{3}$

$\varphi(28, 4) = \varphi(28, 3) - \varphi(4, 3) = 7 - 1 = \mathbf{6}$
$\varphi(28, 3) = \varphi(28, 2) - \varphi(5, 2) = 9 - 2 = 7$
$\varphi(28, 2) = \varphi(28, 1) - \varphi(9, 1) = 14 - 5 = 9$
$\varphi(5, 2) = \varphi(5, 1) - \varphi(1, 1) = 3 - 1 = \mathbf{2}$

$\varphi(23, 5) = \varphi(23, 4) - \varphi(2, 4) = 6 - 1 = \mathbf{5}$
$\varphi(23, 4) = \varphi(23, 3) - \varphi(3, 3) = 7 - 1 = 6$
$\varphi(23, 3) = \varphi(23, 2) - \varphi(4, 2) = 8 - 1 = 7$
$\varphi(23, 2) = \varphi(23, 1) - \varphi(7, 1) = 12 - 4 = 8$

$\varphi(4347, 8) = \varphi(4347, 7) - \varphi(228, 7) = 785 - 43 = \mathbf{742}$
$\varphi(4347, 7) = \varphi(4347, 6) - \varphi(255, 6) = 834 - 49 = 785$
$\varphi(4347, 6) = \varphi(4347, 5) - \varphi(334, 5) = 903 - 69 = 834$
$\varphi(4347, 5) = \varphi(4347, 4) - \varphi(395, 4) = 993 - 90 = 903$
$\varphi(4347, 4) = \varphi(4347, 3) - \varphi(621, 3) = 1159 - 166 = 993$
$\varphi(4347, 3) = \varphi(4347, 2) - \varphi(869, 2) = 1449 - 290 = 1159$
$\varphi(4347, 2) = \varphi(4347, 1) - \varphi(1449, 1) = 2174 - 725 = 1449$
$\varphi(869, 2) = \varphi(869, 1) - \varphi(289,1) = 435 - 145 = \mathbf{290}$

$\varphi(621, 3) = \varphi(621, 2) - \varphi(124,2) = 207 - 41 = \mathbf{164}$
$\varphi(621, 2) = \varphi(621, 1) - \varphi(207,1) = 311 - 104 = 207$
$\varphi(124, 2) = \varphi(124,1) - \varphi(41,1) = 62 - 21 = \mathbf{41}$

$\varphi(395, 4) = \varphi(395, 3) - \varphi(56,3) = 105 - 15 = \mathbf{90}$
$\varphi(395, 3) = \varphi(395, 2) - \varphi(79,2) = 132 - 27 = 105$
$\varphi(395, 2) = \varphi(395, 1) - \varphi(131,1) = 198 - 66 = 132$
$\varphi(79, 2) = \varphi(79,1) - \varphi(26,1) = 40 - 13 = \mathbf{27}$

$\varphi(56, 3) = \varphi(56, 2) - \varphi(11, 2) = 19 - 4 = \mathbf{15}$
$\varphi(56, 2) = \varphi(56, 1) - \varphi(18, 1) = 28 - 9 = 19$
$\varphi(11, 2) = \varphi(11, 1) - \varphi(3,1) = 6 - 2 = \mathbf{4}$

$\varphi(334, 5) = \varphi(334, 4) - \varphi(30,4) = 76 - 7 = \mathbf{69}$
$\varphi(334, 4) = \varphi(334, 3) - \varphi(47,3) = 89 - 13 = 76$
$\varphi(334, 3) = \varphi(334, 2) - \varphi(66,2) = 111 - 22 = 89$
$\varphi(334, 2) = \varphi(334, 1) - \varphi(111,1) = 167 - 56 = 111$
$\varphi(66, 2) = \varphi(66,1) - \varphi(22,1) = 33 - 11 = \mathbf{22}$

$\varphi(47, 3) = \varphi(47,2) - \varphi(9,2) = 16 - 3 = \mathbf{13}$
$\varphi(47, 2) = \varphi(47,1) - \varphi(15,1) = 24 - 8 = 16$
$\varphi(9,2) = \varphi(9,1) = \varphi(3,1) = 5 - 2 = \mathbf{3}$

$\varphi(30, 4) = \varphi(30,3) - \varphi(4,3) = 8 - 1 = \mathbf{7}$
$\varphi(30, 3) = \varphi(30,2) - \varphi(6,2) = 10 - 2 = 8$
$\varphi(30, 2) = \varphi(30,1) - \varphi(10,1) = 15 - 5 = 10$

$\varphi(255, 6) = \varphi(255, 5) - \varphi(19,5) = 53 - 4 = \mathbf{49}$
$\varphi(255, 5) = \varphi(255, 4) - \varphi(23,4) = 59 - 6 = 53$
$\varphi(255, 4) = \varphi(255, 3) - \varphi(36,3) = 68 - 9 = 59$
$\varphi(255, 3) = \varphi(255, 2) - \varphi(51,2) = 85 - 17 = 68$
$\varphi(255, 2) = \varphi(255, 1) - \varphi(85,1) = 128 - 43 = 85$
$\varphi(51, 2) = \varphi(51,1) - \varphi(17,1) = 26 - 9 = \mathbf{17}$

$\varphi(36, 3) = \varphi(36,2) - \varphi(7,2) = 12 - 3 = \mathbf{9}$
$\varphi(36, 2) = \varphi(36,1) - \varphi(12,1) = 18 - 6 = 12$
$\varphi(7,2) = \varphi(7,1) - \varphi(2,1) = 4 - 1 = 3$

$\varphi(23, 4) = \varphi(23,3) - \varphi(3,3) = 7 - 1 = \mathbf{6}$
$\varphi(23, 3) = \varphi(23,2) - \varphi(4,2) = 8 - 1 = 7$
$\varphi(23, 2) = \varphi(23,1) - \varphi(7,1) = 12 - 4 = 8$

$\varphi(19, 5) = \varphi(19,4) - \varphi(1,4) = 5 - 1 = \mathbf{4}$
$\varphi(19, 4) = \varphi(19,3) - \varphi(2,3) = 6 - 1 = 5$
$\varphi(19, 3) = \varphi(19,2) - \varphi(3,2) = 7 - 1 = 6$
$\varphi(19, 2) = \varphi(19,1) - \varphi(6,1) = 10 - 3 = 7$

$\varphi(228, 7) = \varphi(228, 6) - \varphi(13,6) = 44 - 1 = \mathbf{43}$
$\varphi(228, 6) = \varphi(228, 5) - \varphi(17,5) = 47 - 3 = 44$
$\varphi(228, 5) = \varphi(228, 4) - \varphi(20,4) = 52 - 5 = 47$
$\varphi(228, 4) = \varphi(228, 3) - \varphi(32,3) = 61 - 9 = 52$
$\varphi(228, 3) = \varphi(228, 2) - \varphi(45,2) = 76 - 15 = 61$
$\varphi(228, 2) = \varphi(228, 1) - \varphi(76,1) = 114 - 38 = 76$
$\varphi(45, 2) = \varphi(45,1) - \varphi(15,1) = 23 - 8 = \mathbf{15}$

$\varphi(32, 3) = \varphi(32,2) - \varphi(6,2) = 11 - 2 = \mathbf{9}$
$\varphi(32, 2) = \varphi(32,1) - \varphi(10,1) = 16 - 5 = 11$
$\varphi(3448,9) = \varphi(3448,8) - \varphi(149,8) = 584 - 28 = \mathbf{556}$
$\varphi(3448,8) = \varphi(3448,7) - \varphi(181,7) = 620 - 36 = 584$
$\varphi(3448,7) = \varphi(3448,6) - \varphi(202,6) = 661 - 41 = 620$
$\varphi(3448,6) = \varphi(3448,5) - \varphi(265,5) = 716 - 55 = 661$
$\varphi(3448,5) = \varphi(3448,4) - \varphi(313,4) = 788 - 72 = 716$
$\varphi(3448,4) = \varphi(3448,3) - \varphi(492,3) = 919 - 131 = 788$
$\varphi(3448,3) = \varphi(3448,2) - \varphi(689,2) = 1149 - 230 = 919$
$\varphi(3448,2) = \varphi(3448,1) - \varphi(1149,1) = 1724 - 575 = 1149$
$\varphi(689,2) = \varphi(689,1) - \varphi(229,1) = 345 - 115 = \mathbf{230}$

$\varphi(492,3) = \varphi(492,2) - \varphi(98,2) = 164 - 33 = \mathbf{131}$
$\varphi(492,2) = \varphi(492,1) - \varphi(164,1) = 246 - 82 = 164$
$\varphi(98,2) = \varphi(98,1) - \varphi(32,1) = 49 - 16 = \mathbf{33}$

$\varphi(313,4) = \varphi(313,3) - \varphi(44,3) = 84 - 12 = \mathbf{72}$
$\varphi(313,3) = \varphi(313,2) - \varphi(62,2) = 105 - 21 = 84$
$\varphi(313,2) = \varphi(313,1) - \varphi(104,1) = 157 - 52 = 105$
$\varphi(62,2) = \varphi(62,1) - \varphi(20,1) = 31 - 10 = \mathbf{21}$

$\varphi(44,3) = \varphi(44,2) - \varphi(8,2) = 15 - 3 = 12$
$\varphi(44,2) = \varphi(44,1) - \varphi(14,1) = 22 - 7 = 15$

$\varphi(265,5) = \varphi(265,4) - \varphi(24,4) = 61 - 6 = \mathbf{55}$
$\varphi(265,4) = \varphi(265,3) - \varphi(37,3) = 71 - 10 = 61$
$\varphi(265,3) = \varphi(265,2) - \varphi(53,2) = 89 - 18 = 71$
$\varphi(265,2) = \varphi(265,1) - \varphi(88,1) = 133 - 44 = 89$
$\varphi(53,2) = \varphi(53,1) - \varphi(17,1) = 27 - 9 = \mathbf{18}$

$\varphi(37,3) = \varphi(37,2) - \varphi(7,2) = 13 - 3 = \mathbf{10}$
$\varphi(37,2) = \varphi(37,1) - \varphi(12,1) = 19 - 6 = 13$

$\varphi(202,6) = \varphi(202,5) - \varphi(15,5) = 43 - 2 = \mathbf{41}$
$\varphi(202,5) = \varphi(202,4) - \varphi(18,4) = 47 - 4 = 43$
$\varphi(202,4) = \varphi(202,3) - \varphi(28,3) = 54 - 7 = 47$
$\varphi(202,3) = \varphi(202,2) - \varphi(40,2) = 67 - 13 = 54$
$\varphi(202,2) = \varphi(202,1) - \varphi(67,1) = 101 - 34 = 67$

$\varphi(181,7) = \varphi(181,6) - \varphi(10,6) = 37 - 1 = \mathbf{36}$
$\varphi(181,6) = \varphi(181,5) - \varphi(13,5) = 39 - 2 = 37$
$\varphi(181,5) = \varphi(181,4) - \varphi(16,4) = 42 - 3 = 39$
$\varphi(181,4) = \varphi(181,3) - \varphi(25,3) = 49 - 7 = 42$
$\varphi(181,3) = \varphi(181,2) - \varphi(36,2) = 61 - 12 = 49$
$\varphi(181,2) = \varphi(181,1) - \varphi(50,1) = 91 - 30 = 61$
$\varphi(36,2) = \varphi(36,1) - \varphi(12,1) = 18 - 6 = \mathbf{12}$

$\varphi(25,3) = \varphi(25,2) - \varphi(5,2) = 9 - 2 = \mathbf{7}$
$\varphi(25,2) = \varphi(25,1) - \varphi(8,1) = 13 - 4 = 9$

$\varphi(149,8) = \varphi(149,7) - \varphi(7,7) = 29 - 1 = \mathbf{28}$
$\varphi(149,7) = \varphi(149,6) - \varphi(8,6) = 30 - 1 = 29$
$\varphi(149,6) = \varphi(149,5) - \varphi(11,5) = 31 - 1 = 30$
$\varphi(149,5) = \varphi(149,4) - \varphi(13,4) = 34 - 3 = 31$
$\varphi(149,4) = \varphi(149,3) - \varphi(21,3) = 40 - 6 = 34$
$\varphi(149,3) = \varphi(149,2) - \varphi(29,2) = 50 - 10 = 40$
$\varphi(149,2) = \varphi(149,1) - \varphi(49,1) = 75 - 25 = 50$
$\varphi(29,2) = \varphi(29,1) - \varphi(9,1) = 15 - 5 = \mathbf{10}$

$\varphi(3225,10) = \varphi(3225,9) - \varphi(111,9) = 521 - 21 = \mathbf{500}$
$\varphi(3225,9) = \varphi(3225,8) - \varphi(140,8) = 548 - 27 = 521$
$\varphi(3225,8) = \varphi(3225,7) - \varphi(169,7) = 581 - 33 = 548$
$\varphi(3225,7) = \varphi(3225,6) - \varphi(189,6) = 618 - 37 = 581$
$\varphi(3225,6) = \varphi(3225,5) - \varphi(248,5) = 670 - 52 = 618$
$\varphi(3225,5) = \varphi(3225,4) - \varphi(293,4) = 738 - 68 = 670$
$\varphi(3225,4) = \varphi(3225,3) - \varphi(460,3) = 860 - 122 = 738$
$\varphi(3225,3) = \varphi(3225,2) - \varphi(645,2) = 1075 - 215 = 860$
$\varphi(3225,2) = \varphi(3225,1) - \varphi(1075,1) = 1613 - 538 = 1075$
$\varphi(645,2) = \varphi(645,1) - \varphi(215,1) = 323 - 108 = \mathbf{215}$

$\varphi(460,3) = \varphi(460,2) - \varphi(92,2) = 153 - 31 = \mathbf{122}$
$\varphi(460,2) = \varphi(460,1) - \varphi(153,1) = 230 - 77 = 153$
$\varphi(92,2) = \varphi(92,1) - \varphi(30,1) = 46 - 15 = \mathbf{31}$

$\varphi(293,4) = \varphi(293,3) - \varphi(41,3) = 79 - 11 = \mathbf{68}$
$\varphi(293,3) = \varphi(293,2) - \varphi(58,2) = 98 - 19 = 79$
$\varphi(293,2) = \varphi(293,1) - \varphi(97,1) = 147 - 49 = 98$
$\varphi(58,2) = \varphi(58,1) - \varphi(19,1) = 29 - 10 = \mathbf{19}$

$\varphi(41,3) = \varphi(41,2) - \varphi(8,2) = 14 - 3 = \mathbf{11}$
$\varphi(41,2) = \varphi(41,1) - \varphi(13,1) = 21 - 7 = 14$
$\varphi(8,2) = \varphi(8,1) - \varphi(2,1) = 4 - 1 = \mathbf{3}$

$\varphi(248,5) = \varphi(248,4) - \varphi(22,4) = 57 - 5 = \mathbf{52}$
$\varphi(248,4) = \varphi(248,3) - \varphi(35,3) = 66 - 9 = 57$
$\varphi(248,3) = \varphi(248,2) - \varphi(49,2) = 83 - 17 = 66$
$\varphi(248,2) = \varphi(248,1) - \varphi(82,1) = 124 - 41 = 83$
$\varphi(49,2) = \varphi(49,1) - \varphi(16,1) = 25 - 8 = \mathbf{17}$

$\varphi(35,3) = \varphi(35,2) - \varphi(7,2) = 12 - 3 = \mathbf{9}$
$\varphi(35,2) = \varphi(35,1) - \varphi(11,1) = 18 - 6 = 12$
$\varphi(7,2) = \varphi(7,1) - \varphi(2,1) = 4 - 1 = \mathbf{3}$

$\varphi(189,6) = \varphi(189,5) - \varphi(14,5) = 39 - 2 = \mathbf{37}$
$\varphi(189,5) = \varphi(189,4) - \varphi(17,4) = 43 - 4 = 39$
$\varphi(189,4) = \varphi(189,3) - \varphi(27,3) = 50 - 7 = 43$
$\varphi(189,3) = \varphi(189,2) - \varphi(37,2) = 63 - 13 = 50$
$\varphi(189,2) = \varphi(189,1) - \varphi(63,1) = 95 - 32 = 63$
$\varphi(37,2) = \varphi(37,1) - \varphi(12,1) = 19 - 6 = \mathbf{13}$

$\varphi(27,3) = \varphi(27,2) - \varphi(5,2) = 9 - 2 = 7$
$\varphi(27,2) = \varphi(27,1) - \varphi(9,1) = 14 - 5 = 9$
$\varphi(5,2) = \varphi(5,1) - \varphi(2,1) = 3 - 1 = \mathbf{2}$

$\varphi(17,4) = \varphi(17,3) - \varphi(1,3) = 5 - 1 = \mathbf{4}$
$\varphi(17,3) = \varphi(17,2) - \varphi(2,2) = 6 - 1 = 5$
$\varphi(17,2) = \varphi(17,1) - \varphi(5,1) = 9 - 3 = 6$

$\varphi(169,7) = \varphi(169,6) - \varphi(9,6) = 34 - 1 = \mathbf{33}$
$\varphi(169,6) = \varphi(169,5) - \varphi(13,5) = 36 - 2 = 34$
$\varphi(169,5) = \varphi(169,4) - \varphi(15,4) = 39 - 3 = 36$
$\varphi(169,4) = \varphi(169,3) - \varphi(24,3) = 46 - 7 = 39$
$\varphi(169,3) = \varphi(169,2) - \varphi(33,2) = 57 - 11 = 46$
$\varphi(162,2) = \varphi(169,1) - \varphi(56,1) = 85 - 28 = 57$
$\varphi(33,2) = \varphi(33,1) - \varphi(11,1) = 17 - 6 = \mathbf{11}$

$\varphi(24,3) = \varphi(24,2) - \varphi(4,2) = 8 - 1 = 7$
$\varphi(24,2) = \varphi(24,1) - \varphi(8,1) = 12 - 4 = 8$

$\varphi(140,8) = \varphi(140,7) - \varphi(7,7) = 28 - 1 = \mathbf{27}$
$\varphi(140,7) = \varphi(140,6) - \varphi(8,6) = 29 - 1 = 28$
$\varphi(140,6) = \varphi(140,5) - \varphi(10,5) = 30 - 1 = 29$
$\varphi(140,5) = \varphi(140,4) - \varphi(12,4) = 32 - 2 = 30$
$\varphi(140,4) = \varphi(140,3) - \varphi(20,3) = 38 - 6 = 32$
$\varphi(140,3) = \varphi(140,2) - \varphi(28,2) = 47 - 9 = 38$
$\varphi(140,2) = \varphi(140,1) - \varphi(46,1) = 70 - 23 = 47$
$\varphi(28,2) = \varphi(28,1) - \varphi(9,1) = 14 - 5 = 9$

$\varphi(20,3) = \varphi(20,2) - \varphi(4,2) = 7 - 1 = 6$
$\varphi(20,2) = \varphi(20,1) - \varphi(6,1) = 10 - 3 = 7$
$\varphi(4,2) = \varphi(4,1) - \varphi(1,1) = 2 - 1 = \mathbf{1}$

$\varphi(12,4) = \varphi(12,3) - \varphi(1,3) = 3 - 1 = \mathbf{2}$
$\varphi(12,3) = \varphi(12,2) - \varphi(2,2) = 4 - 1 = 3$
$\varphi(12,2) = \varphi(12,1) - \varphi(4,1) = 6 - 2 = 4$

$\varphi(111,9) = \varphi(111,8) - \varphi(4,8) = 22 - 1 = \mathbf{21}$
$\varphi(111,8) = \varphi(111,7) - \varphi(5,7) = 23 - 1 = 22$
$\varphi(111,7) = \varphi(111,6) - \varphi(6,6) = 24 - 1 = 23$
$\varphi(111,6) = \varphi(111,5) - \varphi(8,5) = 25 - 1 = 24$
$\varphi(111,5) = \varphi(111,4) - \varphi(10,4) = 26 - 1 = 25$
$\varphi(111,4) = \varphi(111,3) - \varphi(15,3) = 30 - 4 = 26$
$\varphi(111,3) = \varphi(111,2) - \varphi(22,2) = 37 - 7 = 30$
$\varphi(111,2) = \varphi(111,1) - \varphi(37,1) = 56 - 19 = 37$
$\varphi(22,2) = \varphi(22,1) - \varphi(7,1) = 11 - 4 = \mathbf{7}$

$\varphi(2702,11) = \varphi(2702,10) - \varphi(87,10) = 420 - 14 = \mathbf{406}$
$\varphi(2702,10) = \varphi(2702,9) - \varphi(93,9) = 436 - 16 = 420$
$\varphi(2702,9) = \varphi(2702,8) - \varphi(117,8) = 459 - 23 = 436$
$\varphi(2702,8) = \varphi(2702,7) - \varphi(142,7) = 487 - 28 = 459$
$\varphi(2702,7) = \varphi(2702,6) - \varphi(158,6) = 519 - 32 = 487$
$\varphi(2702,6) = \varphi(2702,5) - \varphi(207,5) = 562 - 43 = 519$
$\varphi(2702,5) = \varphi(2702,4) - \varphi(245,4) = 618 - 56 = 562$
$\varphi(2702,4) = \varphi(2702,3) - \varphi(386,3) = 721 - 103 = 618$
$\varphi(2702,3) = \varphi(2702,2) - \varphi(540,2) = 901 - 180 = 721$
$\varphi(2702,2) = \varphi(2702,1) - \varphi(900,1) = 1351 - 450 = 901$
$\varphi(540,2) = \varphi(540,1) - \varphi(180,1) = 270 - 90 = \mathbf{180}$

$\varphi(386,3) = \varphi(386,2) - \varphi(77,2) = 129 - 26 = \mathbf{103}$
$\varphi(386,2) = \varphi(386,1) - \varphi(128,1) = 193 - 64 = 129$
$\varphi(77,2) = \varphi(77,1) - \varphi(25,1) = 39 - 13 = \mathbf{26}$

$\varphi(245,4) = \varphi(245,3) - \varphi(35,3) = 65 - 9 = \mathbf{56}$
$\varphi(245,3) = \varphi(245,2) - \varphi(49,2) = 82 - 17 = 65$
$\varphi(245,2) = \varphi(245,1) - \varphi(81,1) = 123 - 41 = 82$
$\varphi(49,2) = \varphi(49,1) - \varphi(16,1) = 25 - 8 = \mathbf{17}$

$\varphi(35,3) = \varphi(35,2) - \varphi(7,2) = 12 - 3 = \mathbf{9}$
$\varphi(35,2) = \varphi(35,1) - \varphi(11,1) = 18 - 6 = 12$
$\varphi(7,2) = \varphi(7,1) - \varphi(2,1) = 4 - 1 = \mathbf{3}$

$\varphi(207,5) = \varphi(207,4) - \varphi(18,4) = 47 - 4 = \mathbf{43}$
$\varphi(207,4) = \varphi(207,3) - \varphi(29,3) = 55 - 8 = 47$
$\varphi(207,3) = \varphi(207,2) - \varphi(41,2) = 69 - 14 = 55$
$\varphi(207,2) = \varphi(207,1) - \varphi(69,1) = 104 - 35 = 69$
$\varphi(41,2) = \varphi(41,1) - \varphi(13,1) = 21 - 7 = \mathbf{14}$

$\varphi(29,3) = \varphi(29,2) - \varphi(5,2) = 10 - 2 = 8$
$\varphi(29,2) = \varphi(29,1) - \varphi(9,1) = 15 - 5 = 10$
$\varphi(5,2) = \varphi(5,1) - \varphi(1,1) = 3 - 1 = 2$

$\varphi(18,4) = \varphi(18,3) - \varphi(2,3) = 5 - 1 = \mathbf{4}$
$\varphi(18,3) = \varphi(18,2) - \varphi(3,2) = 6 - 1 = 5$
$\varphi(18,2) = \varphi(18,1) - \varphi(6,1) = 9 - 3 = 6$

$\varphi(158,6) = \varphi(158,5) - \varphi(12,5) = 33 - 1 = \mathbf{32}$
$\varphi(158,5) = \varphi(158,4) - \varphi(14,4) = 36 - 3 = 33$
$\varphi(158,4) = \varphi(158,3) - \varphi(22,3) = 42 - 6 = 36$

$\varphi(158, 3) = \varphi(158, 2) - \varphi(31, 2) = 53 - 11 = 42$
$\varphi(158, 2) = \varphi(158, 1) - \varphi(52, 1) = 79 - 26 = 53$
$\varphi(31, 2) = \varphi(31, 1) - \varphi(10,1) = 16 - 5 = 11$

$\varphi(22, 3) = \varphi(22, 2) - \varphi(4,2) = 7 - 1 = 6$
$\varphi(22, 2) = \varphi(22, 1) - \varphi(7,1) = 11 - 4 = 7$
$\varphi(4, 2) = \varphi(4, 1) - \varphi(1, 1) = 2 - 1 = 1$

$\varphi(14, 4) = \varphi(14, 3) - \varphi(2,3) = 4 - 1 = 3$
$\varphi(14, 3) = \varphi(14, 2) - \varphi(2,2) = 5 - 1 = 4$
$\varphi(14, 2) = \varphi(14, 1) - \varphi(4,1) = 7 - 2 = 5$

$\varphi(142, 7) = \varphi(142, 6) - \varphi(8, 6) = 29 - 1 = \mathbf{28}$
$\varphi(142, 6) = \varphi(142, 5) - \varphi(10, 5) = 30 - 1 = 29$
$\varphi(142, 5) = \varphi(142, 4) - \varphi(12, 4) = 32 - 2 = 30$
$\varphi(142, 4) = \varphi(142, 3) - \varphi(20, 3) = 38 - 6 = 32$
$\varphi(142, 3) = \varphi(142, 2) - \varphi(28, 2) = 47 - 9 = 38$
$\varphi(142, 2) = \varphi(142, 1) - \varphi(47, 1) = 71 - 24 = 47$
$\varphi(28, 2) = \varphi(28, 1) - \varphi(9,1) = 14 - 5 = 9$

$\varphi(20, 3) = \varphi(20, 2) - \varphi(4,2) = 7 - 1 = \mathbf{6}$
$\varphi(20, 2) = \varphi(20, 1) - \varphi(6,1) = 10 - 3 = 7$
$\varphi(4,2) = \varphi(4, 1) - \varphi(1,1) = 2 - 1 = 1$

$\varphi(12, 4) = \varphi(12, 3) - \varphi(1,3) = 3 - 1 = \mathbf{2}$
$\varphi(12, 3) = \varphi(12, 2) - \varphi(2,2) = 4 - 1 = 3$
$\varphi(12, 2) = \varphi(12, 1) - \varphi(4,1) = 6 - 2 = 4$

$\varphi(117, 8) = \varphi(117, 7) - \varphi(6, 7) = 24 - 1 = 23$
$\varphi(117, 7) = \varphi(117, 6) - \varphi(6, 6) = 25 - 1 = 24$
$\varphi(117, 6) = \varphi(117, 5) - \varphi(9, 5) = 26 - 1 = 25$
$\varphi(117, 5) = \varphi(117, 4) - \varphi(10, 4) = 27 - 1 = 26$
$\varphi(117, 4) = \varphi(117, 3) - \varphi(16, 3) = 31 - 4 = 27$
$\varphi(117, 3) = \varphi(117, 2) - \varphi(23, 2) = 39 - 8 = 31$
$\varphi(117, 2) = \varphi(117, 1) - \varphi(39, 1) = 59 - 20 = 39$
$\varphi(23, 2) = \varphi(23, 1) - \varphi(7,1) = 12 - 4 = 8$

$\varphi(16, 3) = \varphi(16, 2) - \varphi(3,2) = 5 - 1 = 4$
$\varphi(16, 2) = \varphi(16, 1) - \varphi(5,1) = 8 - 3 = 5$

$\varphi(93, 9) = \varphi(93, 8) - \varphi(4, 8) = 17 - 1 = \mathbf{16}$
$\varphi(93, 8) = \varphi(93, 7) - \varphi(4, 7) = 18 - 1 = 17$
$\varphi(93, 7) = \varphi(93, 6) - \varphi(5, 6) = 19 - 1 = 18$

$\varphi(93,6) = \varphi(93,5) - \varphi(7,5) = 20 - 1 = 19$
$\varphi(93,5) = \varphi(93,4) - \varphi(8,4) = 21 - 1 = 20$
$\varphi(93,4) = \varphi(93,3) - \varphi(13,3) = 25 - 4 = 21$
$\varphi(93,3) = \varphi(93,2) - \varphi(18,2) = 31 - 6 = 25$
$\varphi(93,2) = \varphi(93,1) - \varphi(31,1) = 47 - 16 = 31$
$\varphi(18,2) = \varphi(18,1) - \varphi(6,1) = 9 - 3 = \mathbf{6}$

$\varphi(13,3) = \varphi(13,2) - \varphi(2,2) = 5 - 1 = \mathbf{4}$
$\varphi(13,2) = \varphi(13,1) - \varphi(4,1) = 7 - 2 = 5$

$\varphi(87,10) = \varphi(87,9) - \varphi(3,9) = 15 - 1 = 14$
$\varphi(87,9) = \varphi(87,8) - \varphi(3,8) = 16 - 1 = 15$
$\varphi(87,8) = \varphi(87,7) - \varphi(4,7) = 17 - 1 = 16$
$\varphi(87,7) = \varphi(87,6) - \varphi(5,6) = 18 - 1 = 17$
$\varphi(87,6) = \varphi(87,5) - \varphi(6,5) = 19 - 1 = 18$
$\varphi(87,5) = \varphi(87,4) - \varphi(7,4) = 20 - 1 = 19$
$\varphi(87,4) = \varphi(87,3) - \varphi(12,3) = 23 - 3 = 20$
$\varphi(87,3) = \varphi(87,2) - \varphi(17,2) = 29 - 6 = 23$
$\varphi(87,2) = \varphi(87,1) - \varphi(29,1) = 44 - 15 = 29$
$\varphi(17,2) = \varphi(17,1) - \varphi(5,1) = 9 - 3 = \mathbf{6}$

$\varphi(12,3) = \varphi(12,2) - \varphi(2,2) = 4 - 1 = \mathbf{3}$
$\varphi(12,2) = \varphi(12,1) - \varphi(4,1) = 6 - 2 = 4$

$\varphi(2439,12) = \varphi(2439,11) - \varphi(65,11) = 367 - 8 = \mathbf{359}$
$\varphi(2439,11) = \varphi(2439,10) - \varphi(78,10) = 379 - 12 = 367$
$\varphi(2439,10) = \varphi(2439,9) - \varphi(84,9) = 394 - 15 = 379$
$\varphi(2439,9) = \varphi(2439,8) - \varphi(106,8) = 414 - 20 = 394$
$\varphi(2439,8) = \varphi(2439,7) - \varphi(128,7) = 439 - 25 = 414$
$\varphi(2439,7) = \varphi(2439,6) - \varphi(143,6) = 468 - 29 = 439$
$\varphi(2439,6) = \varphi(2439,5) - \varphi(187,5) = 507 - 39 = 468$
$\varphi(2439,5) = \varphi(2439,4) - \varphi(221,4) = 557 - 50 = 507$
$\varphi(2439,4) = \varphi(2439,3) - \varphi(348,3) = 650 - 93 = 557$
$\varphi(2439,3) = \varphi(2439,2) - \varphi(487,2) = 813 - 163 = 650$
$\varphi(2439,2) = \varphi(2439,1) - \varphi(813,1) = 1220 - 407 = 813$
$\varphi(487,2) = \varphi(487,1) - \varphi(162,1) = 244 - 81 = 163$

$\varphi(348,3) = \varphi(348,2) - \varphi(69,2) = 116 - 23 = 93$
$\varphi(348,2) = \varphi(348,1) - \varphi(116,1) = 174 - 58 = 116$
$\varphi(69,2) = \varphi(69,1) - \varphi(23,1) = 35 - 12 = \mathbf{23}$

$\varphi(221, 4) = \varphi(221, 3) - \varphi(31, 3) = 59 - 9 = \mathbf{50}$
$\varphi(221, 3) = \varphi(221, 2) - \varphi(44, 2) = 74 - 15 = 59$
$\varphi(221, 2) = \varphi(221, 1) - \varphi(73, 1) = 111 - 37 = 74$
$\varphi(44, 2) = \varphi(44, 1) - \varphi(14,1) = 22 - 7 = \mathbf{15}$

$\varphi(31, 3) = \varphi(31, 2) - \varphi(6,2) = 11 - 2 = 9$
$\varphi(31, 2) = \varphi(31, 1) - \varphi(10,1) = 16 - 5 = 11$
$\varphi(6, 2) = \varphi(6, 1) - \varphi(2,1) = 3 - 1 = \mathbf{2}$

$\varphi(187, 5) = \varphi(187, 4) - \varphi(17, 4) = 43 - 4 = \mathbf{39}$
$\varphi(187, 4) = \varphi(187, 3) - \varphi(26, 3) = 50 - 7 = 43$
$\varphi(187, 3) = \varphi(187, 2) - \varphi(37, 2) = 63 - 13 = 50$
$\varphi(187, 2) = \varphi(187, 1) - \varphi(62, 1) = 94 - 31 = 63$
$\varphi(37, 2) = \varphi(37, 1) - \varphi(12,1) = 19 - 6 = \mathbf{13}$

$\varphi(26, 3) = \varphi(26, 2) - \varphi(5, 2) = 9 - 2 = \mathbf{7}$
$\varphi(26, 2) = \varphi(26, 1) - \varphi(8, 1) = 13 - 4 = 9$
$\varphi(5, 2) = \varphi(5, 1) - \varphi(1, 1) = 3 - 1 = 2$

$\varphi(17, 4) = \varphi(17, 3) - \varphi(2, 3) = 5 - 1 = 4$
$\varphi(17, 3) = \varphi(17, 2) - \varphi(3, 2) = 6 - 1 = 5$
$\varphi(17, 2) = \varphi(17, 1) - \varphi(5, 1) = 9 - 3 = 6$

$\varphi(143, 6) = \varphi(143, 5) - \varphi(11,5) = 30 - 1 = \mathbf{29}$
$\varphi(143, 5) = \varphi(143, 4) - \varphi(13,4) = 33 - 3 = 30$
$\varphi(143, 4) = \varphi(143, 3) - \varphi(20,3) = 39 - 6 = 33$
$\varphi(143, 3) = \varphi(143, 2) - \varphi(28,2) = 48 - 9 = 39$
$\varphi(143, 2) = \varphi(143, 1) - \varphi(47,1) = 72 - 24 = 48$
$\varphi(28, 2) = \varphi(28, 1) - \varphi(9,1) = 14 - 5 = 9$

$\varphi(20, 3) = \varphi(20, 2) - \varphi(4,2) = 7 - 1 = \mathbf{6}$
$\varphi(20, 2) = \varphi(20, 1) - \varphi(6, 1) = 10 - 3 = 7$
$\varphi(4, 2) = \varphi(4, 1) - \varphi(1, 1) = 2 - 1 = \mathbf{1}$

$\varphi(13, 4) = \varphi(13, 3) - \varphi(1,3) = 4 - 1 = \mathbf{3}$
$\varphi(13, 3) = \varphi(13, 2) - \varphi(2,2) = 5 - 1 = 4$
$\varphi(13, 2) = \varphi(13, 1) - \varphi(4,1) = 7 - 2 = 5$

$\varphi(128, 7) = \varphi(128, 6) - \varphi(7, 6) = 26 - 1 = \mathbf{25}$
$\varphi(128, 6) = \varphi(128, 5) - \varphi(9, 5) = 27 - 1 = 26$
$\varphi(128, 5) = \varphi(128, 4) - \varphi(11, 4) = 29 - 2 = 27$
$\varphi(128, 4) = \varphi(128, 3) - \varphi(18, 3) = 34 - 5 = 29$

$\varphi(128,3) = \varphi(128,2) - \varphi(25,2) = 43 - 9 = 34$
$\varphi(128,2) = \varphi(128,1) - \varphi(42,1) = 64 - 21 = 43$
$\varphi(25,2) = \varphi(25,1) - \varphi(8,1) = 13 - 4 = 9$

$\varphi(18,3) = \varphi(18,2) - \varphi(3,2) = 6 - 1 = 5$
$\varphi(18,2) = \varphi(18,1) - \varphi(6,1) = 9 - 3 = 6$

$\varphi(11,4) = \varphi(11,3) - \varphi(1,3) = 3 - 1 = 2$
$\varphi(11,3) = \varphi(11,2) - \varphi(2,2) = 4 - 1 = 3$
$\varphi(11,2) = \varphi(11,1) - \varphi(3,1) = 6 - 2 = 4$

$\varphi(106,8) = \varphi(106,7) - \varphi(5,7) = 21 - 1 = \mathbf{20}$
$\varphi(106,7) = \varphi(106,6) - \varphi(6,6) = 22 - 1 = 21$
$\varphi(106,6) = \varphi(106,5) - \varphi(8,5) = 23 - 1 = 22$
$\varphi(106,5) = \varphi(106,4) - \varphi(9,4) = 24 - 1 = 23$
$\varphi(106,4) = \varphi(106,3) - \varphi(15,3) = 28 - 4 = 24$
$\varphi(106,3) = \varphi(106,2) - \varphi(21,2) = 35 - 7 = 28$
$\varphi(106,2) = \varphi(106,1) - \varphi(35,1) = 53 - 18 = 35$
$\varphi(21,2) = \varphi(21,1) - \varphi(7,1) = 11 - 4 = 7$

$\varphi(15,3) = \varphi(15,2) - \varphi(3,2) = 5 - 1 = 4$
$\varphi(15,2) = \varphi(15,1) - \varphi(5,1) = 8 - 3 = 5$

$\varphi(84,9) = \varphi(84,8) - \varphi(3,8) = 16 - 1 = \mathbf{15}$
$\varphi(84,8) = \varphi(84,7) - \varphi(4,7) = 17 - 1 = 16$
$\varphi(84,7) = \varphi(84,6) - \varphi(4,6) = 18 - 1 = 17$
$\varphi(84,6) = \varphi(84,5) - \varphi(6,5) = 19 - 1 = 18$
$\varphi(84,5) = \varphi(84,4) - \varphi(7,4) = 20 - 1 = 19$
$\varphi(84,4) = \varphi(84,3) - \varphi(12,3) = 23 - 3 = 20$
$\varphi(84,3) = \varphi(84,2) - \varphi(16,2) = 28 - 5 = 23$
$\varphi(84,2) = \varphi(84,1) - \varphi(28,1) = 42 - 14 = 28$
$\varphi(16,2) = \varphi(16,1) - \varphi(5,1) = 8 - 3 = \mathbf{5}$

$\varphi(12,3) = \varphi(12,2) - \varphi(2,2) = 4 - 1 = \mathbf{3}$
$\varphi(12,2) = \varphi(12,1) - \varphi(4,1) = 6 - 2 = 4$

$\varphi(78,10) = \varphi(78,9) - \varphi(2,9) = 13 - 1 = \mathbf{12}$
$\varphi(78,9) = \varphi(78,8) - \varphi(3,8) = 14 - 1 = 13$
$\varphi(78,8) = \varphi(78,7) - \varphi(4,7) = 15 - 1 = 14$
$\varphi(78,7) = \varphi(78,6) - \varphi(4,6) = 16 - 1 = 15$
$\varphi(78,6) = \varphi(78,5) - \varphi(6,5) = 17 - 1 = 16$
$\varphi(78,5) = \varphi(78,4) - \varphi(7,4) = 18 - 1 = 17$

$\varphi(78,4) = \varphi(78,3) - \varphi(11,3) = 21 - 3 = 18$
$\varphi(78,3) = \varphi(78,2) - \varphi(15,2) = 26 - 5 = 21$
$\varphi(78,2) = \varphi(78,1) - \varphi(26,1) = 39 - 13 = 26$
$\varphi(15,2) = \varphi(15,1) - \varphi(5,1) = 8 - 3 = \mathbf{5}$

$\varphi(11,3) = \varphi(11,2) - \varphi(2,2) = 4 - 1 = \mathbf{3}$
$\varphi(11,2) = \varphi(11,1) - \varphi(3,1) = 6 - 2 = 4$

$\varphi(65,11) = \varphi(65,10) - \varphi(2,10) = 9 - 1 = \mathbf{8}$
$\varphi(65,10) = \varphi(65,9) - \varphi(2,9) = 10 - 1 = 9$
$\varphi(65,9) = \varphi(65,8) - \varphi(2,8) = 11 - 1 = 10$
$\varphi(65,8) = \varphi(65,7) - \varphi(3,7) = 12 - 1 = 11$
$\varphi(65,7) = \varphi(65,6) - \varphi(3,6) = 13 - 1 = 12$
$\varphi(65,6) = \varphi(65,5) - \varphi(5,5) = 14 - 1 = 13$
$\varphi(65,5) = \varphi(65,4) - \varphi(5,4) = 15 - 1 = 14$
$\varphi(65,4) = \varphi(65,3) - \varphi(9,3) = 17 - 2 = 15$
$\varphi(65,3) = \varphi(65,2) - \varphi(13,2) = 22 - 5 = 17$
$\varphi(65,2) = \varphi(65,1) - \varphi(21,1) = 33 - 11 = 22$
$\varphi(13,2) = \varphi(13,1) - \varphi(4,1) = 7 - 2 = \mathbf{5}$

$\varphi(9,3) = \varphi(9,2) - \varphi(1,2) = 3 - 1 = \mathbf{2}$
$\varphi(9,2) = \varphi(9,1) - \varphi(3,1) = 5 - 2 = 3$

$\varphi(2325,13) = \varphi(2325,12) - \varphi(56,12) = 341 - 5 = \mathbf{336}$
$\varphi(2325,12) = \varphi(2325,11) - \varphi(62,11) = 349 - 8 = 341$
$\varphi(2325,11) = \varphi(2325,10) - \varphi(75,10) = 361 - 12 = 349$
$\varphi(2325,10) = \varphi(2325,9) - \varphi(80,9) = 375 - 14 = 361$
$\varphi(2325,9) = \varphi(2325,8) - \varphi(101,8) = 394 - 19 = 375$
$\varphi(2325,8) = \varphi(2325,7) - \varphi(122,7) = 418 - 24 = 394$
$\varphi(2325,7) = \varphi(2325,6) - \varphi(136,6) = 445 - 27 = 418$
$\varphi(2325,6) = \varphi(2325,5) - \varphi(178,5) = 482 - 37 = 445$
$\varphi(2325,5) = \varphi(2325,4) - \varphi(211,4) = 531 - 49 = 482$
$\varphi(2325,4) = \varphi(2325,3) - \varphi(332,3) = 620 - 89 = 531$
$\varphi(2325,3) = \varphi(2325,2) - \varphi(465,2) = 775 - 155 = 620$
$\varphi(2325,2) = \varphi(2325,1) - \varphi(775,1) = 1163 - 388 = 775$
$\varphi(465,2) = \varphi(465,1) - \varphi(155,1) = 233 - 78 = 155$

$\varphi(332,3) = \varphi(332,1) = \varphi(66,2) = 111 - 22 = \mathbf{89}$
$\varphi(332,2) = \varphi(332,1) = \varphi(110,1) = 166 - 55 = 111$
$\varphi(66,2) = \varphi(66,1) - \varphi(22,1) = 33 - 11 = 22$

$\varphi(211, 4) = \varphi(211, 3) - \varphi(30, 3) = 57 - 8 = \mathbf{49}$
$\varphi(211, 3) = \varphi(211, 2) - \varphi(42, 2) = 71 - 14 = 57$
$\varphi(211, 2) = \varphi(211, 1) - \varphi(70, 1) = 106 - 35 = 71$
$\varphi(42, 2) = \varphi(42, 1) - \varphi(14, 1) = 21 - 7 = 14$

$\varphi(30, 3) = \varphi(30, 2) - \varphi(6, 2) = 10 - 2 = \mathbf{8}$
$\varphi(30, 2) = \varphi(30, 1) - \varphi(10, 1) = 15 - 5 = 10$
$\varphi(6, 2) = \varphi(6, 1) - \varphi(2, 1) = 3 - 1 = 2$

$\varphi(178, 5) = \varphi(178, 4) - \varphi(16, 4) = 40 - 3 = \mathbf{37}$
$\varphi(178, 4) = \varphi(178, 3) - \varphi(25, 3) = 47 - 7 = 40$
$\varphi(178, 3) = \varphi(178, 2) - \varphi(35, 2) = 59 - 12 = 47$
$\varphi(178, 2) = \varphi(178, 1) - \varphi(59, 1) = 89 - 30 = 59$
$\varphi(35, 2) = \varphi(35, 1) - \varphi(11, 1) = 18 - 6 = \mathbf{12}$

$\varphi(25, 3) = \varphi(25, 2) - \varphi(5, 2) = 9 - 2 = \mathbf{7}$
$\varphi(25, 2) = \varphi(25, 1) - \varphi(8, 1) = 13 - 4 = 9$
$\varphi(5, 2) = \varphi(5, 1) - \varphi(1, 1) = 3 - 1 = 2$

$\varphi(16, 4) = \varphi(16, 3) - \varphi(2, 3) = 4 - 1 = \mathbf{3}$
$\varphi(16, 3) = \varphi(16, 2) - \varphi(3, 2) = 5 - 1 = 4$
$\varphi(16, 2) = \varphi(16, 1) - \varphi(5, 1) = 8 - 3 = 5$

$\varphi(136, 6) = \varphi(136, 5) - \varphi(10, 5) = 28 - 1 = \mathbf{27}$
$\varphi(136, 5) = \varphi(136, 4) - \varphi(12, 4) = 30 - 2 = 28$
$\varphi(136, 4) = \varphi(136, 3) - \varphi(19, 3) = 36 - 6 = 30$
$\varphi(136, 3) = \varphi(136, 2) - \varphi(27, 2) = 45 - 9 = 36$
$\varphi(136, 2) = \varphi(136, 1) - \varphi(45, 1) = 68 - 23 = 45$
$\varphi(27, 2) = \varphi(27, 1) - \varphi(9, 1) = 14 - 5 = 9$
$\varphi(19, 3) = \varphi(19, 2) - \varphi(3, 2) = 7 - 1 = \mathbf{6}$
$\varphi(19, 2) = \varphi(19, 1) - \varphi(6, 1) = 10 - 3 = 7$

$\varphi(12, 4) = \varphi(12, 3) - \varphi(1, 3) = 3 - 1 = \mathbf{2}$
$\varphi(12, 3) = \varphi(12, 2) - \varphi(2, 2) = 4 - 1 = 3$
$\varphi(12, 2) = \varphi(12, 1) - \varphi(4, 1) = 6 - 2 = 4$

$\varphi(122, 7) = \varphi(122, 6) - \varphi(7, 6) = 25 - 1 = \mathbf{24}$
$\varphi(122, 6) = \varphi(122, 5) - \varphi(9, 5) = 26 - 1 = 25$
$\varphi(122, 5) = \varphi(122, 4) - \varphi(11, 4) = 28 - 2 = 26$
$\varphi(122, 4) = \varphi(122, 3) - \varphi(17, 3) = 33 - 5 = 28$
$\varphi(122, 3) = \varphi(122, 2) - \varphi(24, 2) = 41 - 8 = 33$
$\varphi(122, 2) = \varphi(122, 1) - \varphi(40, 1) = 61 - 20 = 41$
$\varphi(24, 2) = \varphi(24, 1) - \varphi(8, 1) = 12 - 4 = \mathbf{8}$

$\varphi(17,3) = \varphi(17,2) - \varphi(3,2) = 6 - 1 = \mathbf{5}$
$\varphi(17,2) = \varphi(17,1) - \varphi(5,1) = 9 - 3 = 6$

$\varphi(11,4) = \varphi(11,3) - \varphi(1,3) = 3 - 1 = 2$
$\varphi(11,3) = \varphi(11,2) - \varphi(2,2) = 4 - 1 = 3$
$\varphi(11,2) = \varphi(11,1) - \varphi(3,1) = 6 - 2 = 4$

$\varphi(101,8) = \varphi(101,7) - \varphi(5,7) = 20 - 1 = \mathbf{19}$
$\varphi(101,7) = \varphi(101,6) - \varphi(5,6) = 21 - 1 = 20$
$\varphi(101,6) = \varphi(101,5) - \varphi(7,5) = 22 - 1 = 21$
$\varphi(101,5) = \varphi(101,4) - \varphi(9,4) = 23 - 1 = 22$
$\varphi(101,4) = \varphi(101,3) - \varphi(14,3) = 27 - 4 = 23$
$\varphi(101,3) = \varphi(101,2) - \varphi(20,2) = 34 - 7 = 27$
$\varphi(101,2) = \varphi(101,1) - \varphi(33,1) = 51 - 17 = 34$
$\varphi(20,2) = \varphi(20,1) - \varphi(6,1) = 10 - 3 = \mathbf{7}$

$\varphi(14,3) = \varphi(14,2) - \varphi(2,2) = 5 - 1 = \mathbf{4}$
$\varphi(14,2) = \varphi(14,1) - \varphi(4,1) = 7 - 2 = 5$

$\varphi(80,9) = \varphi(80,8) - \varphi(3,8) = 15 - 1 = \mathbf{14}$
$\varphi(80,8) = \varphi(80,7) - \varphi(4,7) = 16 - 1 = 15$
$\varphi(80,7) = \varphi(80,6) - \varphi(4,6) = 17 - 1 = 16$
$\varphi(80,6) = \varphi(80,5) - \varphi(6,5) = 18 - 1 = 17$
$\varphi(80,5) = \varphi(80,4) - \varphi(7,4) = 19 - 1 = 18$
$\varphi(80,4) = \varphi(80,3) - \varphi(11,3) = 22 - 3 = 19$
$\varphi(80,3) = \varphi(80,2) - \varphi(16,2) = 27 - 5 = 22$
$\varphi(80,2) = \varphi(80,1) - \varphi(26,1) = 40 - 13 = 27$
$\varphi(16,2) = \varphi(16,1) - \varphi(5,1) = 8 - 3 = \mathbf{5}$

$\varphi(11,3) = \varphi(11,2) - \varphi(2,2) = 4 - 1 = \mathbf{3}$
$\varphi(11,2) = \varphi(11,1) - \varphi(3,1) = 6 - 2 = 4$

$\varphi(75,10) = \varphi(75,9) - \varphi(2,9) = 13 - 1 = \mathbf{12}$
$\varphi(75,9) = \varphi(75,8) - \varphi(3,8) = 14 - 1 = 13$
$\varphi(75,8) = \varphi(75,7) - \varphi(3,7) = 15 - 1 = 14$
$\varphi(75,7) = \varphi(75,6) - \varphi(4,6) = 16 - 1 = 15$
$\varphi(75,6) = \varphi(75,5) - \varphi(5,5) = 17 - 1 = 16$
$\varphi(75,5) = \varphi(75,4) - \varphi(6,4) = 18 - 1 = 17$
$\varphi(75,4) = \varphi(75,3) - \varphi(10,3) = 20 - 2 = 18$
$\varphi(75,3) = \varphi(75,2) - \varphi(15,2) = 25 - 5 = 20$
$\varphi(75,2) = \varphi(75,1) - \varphi(25,1) = 38 - 13 = 25$
$\varphi(15,2) = \varphi(15,1) - \varphi(5,1) = 8 - 3 = 5$

$$\varphi(10, 3) = \varphi(10, 2) - \varphi(2, 2) = 3 - 1 = \mathbf{2}$$
$$\varphi(10, 2) = \varphi(10, 1) - \varphi(3, 1) = 5 - 2 = 3$$

$$\varphi(62,11) = \varphi(62,10) - \varphi(2,10) = 9 - 1 = \mathbf{8}$$
$$\varphi(62,10) = \varphi(62, 9) - \varphi(2, 9) = 10 - 1 = 9$$
$$\varphi(62, 9) = \varphi(62, 8) - \varphi(2, 8) = 11 - 1 = 10$$
$$\varphi(62, 8) = \varphi(62, 7) - \varphi(3, 7) = 12 - 1 = 11$$
$$\varphi(62, 7) = \varphi(62, 6) - \varphi(3, 6) = 13 - 1 = 12$$
$$\varphi(62, 6) = \varphi(62, 5) - \varphi(4, 5) = 14 - 1 = 13$$
$$\varphi(62, 5) = \varphi(62, 4) - \varphi(5, 4) = 15 - 1 = 14$$
$$\varphi(62, 4) = \varphi(62, 3) - \varphi(8, 3) = 17 - 2 = 15$$
$$\varphi(62, 3) = \varphi(62, 2) - \varphi(12, 2) = 21 - 4 = 17$$
$$\varphi(62, 2) = \varphi(62, 1) - \varphi(20, 1) = 31 - 10 = 21$$
$$\varphi(12, 2) = \varphi(12, 1) - \varphi(4,1) = 6 - 2 = \mathbf{4}$$

$$\varphi(8, 3) = \varphi(8, 2) - \varphi(1,2) = 3 - 1 = \mathbf{2}$$
$$\varphi(8, 2) = \varphi(8, 1) - \varphi(2,1) = 4 - 1 = 3$$

$$\varphi(56,12) = \varphi(56,11) - \varphi(1,11) = 6 - 1 = \mathbf{5}$$
$$\varphi(56,11) = \varphi(56,10) - \varphi(1,10) = 7 - 1 = 6$$
$$\varphi(56,10) = \varphi(56, 9) - \varphi(1, 9) = 8 - 1 = 7$$
$$\varphi(56, 9) = \varphi(56, 8) - \varphi(2, 8) = 9 - 1 = 8$$
$$\varphi(56, 8) = \varphi(56, 7) - \varphi(2, 7) = 10 - 1 = 9$$
$$\varphi(56, 7) = \varphi(56, 6) - \varphi(3, 6) = 11 - 1 = 10$$
$$\varphi(56, 6) = \varphi(56, 5) - \varphi(4, 5) = 12 - 1 = 11$$
$$\varphi(56, 5) = \varphi(56, 4) - \varphi(5, 4) = 13 - 1 = 12$$
$$\varphi(56, 4) = \varphi(56, 3) - \varphi(8, 3) = 15 - 2 = 13$$
$$\varphi(56, 3) = \varphi(56, 2) - \varphi(11, 2) = 19 - 4 = 15$$
$$\varphi(56, 2) = \varphi 56, 1) - \varphi(18, 1) = 28 - 9 = 19$$

Obliczenie podług tej metody dla $n = 1{,}000\,000$, wymaga miejsca i czasu prawie 20 razy tyle, co dla $n = 100\,000$; gdyż $\sqrt[3]{n} = 100$, a $\psi(100) = 25$; $\varphi(n, m) = \varphi(1{,}000\,000, 25)$. Ilość więc potrzebnych dzieleń i odejmowań bardzo się powiększa.

Skraca się nieco rachunek powyższy przez uwzględnienie koła luk w numeracyi, jakie powstają przez wyjęcie liczb podzielnych przez liczby pierwsze. Koła te tworzą się podług wzoru

$$\varphi(p_1 \cdot p_2 \cdot \ldots \cdot p_n, n) = (p_1 - 1)(p_2 - 1)(\ldots)(p_n - 1),$$

czyli $\varphi(2, 1) = 2-1 = 1$

$2.3 = 6$; $\varphi(6, 2) = (2-1)(3-1) = 2$

$2.3.5. = 30$; $\varphi(30,3) = 2(5-1) = 8$; $\varphi(60,3) = 16$;
$\varphi(90, 3) = 24$; . . .

$2.3.5.7 = 210$; $\varphi(210,4) = 8(7-1) = 48$; $\varphi(420,4) = 96$;
$\varphi(630, 4) = 144$ . . .

$2.3.5.7.11 = 2310$; $\varphi(2310,5) = 48(11-1) = 480$; $\varphi(4620,5) =$
$= 960$; $\varphi(6930, 5) = 1440$ . . .

$2.3.5.7.11.13 = 30030$; $\varphi(30030,6) = 480(13-1) = 5760$;
$\varphi(60060,6) = 11520$, . . .

$2.3.5.7.11.13.17 = 510510$; $\varphi(510510, 7) = 5760(17-1) = 92160$ i t. d.

Odkrycie tych kół przypisują Drowi Meisselowi. To tylko mię dziwi, że Wertheim, wykładając wyżej przytoczoną formułę Meissela, nic o tych kołach nie wspomina. Dr. Hossfeld, gdym mu je zakomunikował, z początku uznał rzecz za nową, a potem doniósł, że Dr. Meissel je wynalazł i opisał w „Mathematische Annalen“ Bd. II, około roku 1870.

Większe bez porównania od tych kół ułatwienie obliczeń daje tablica analityczna funkcyi $\varphi(m)$. Przytoczę przykład z tablicy z rachunkiem luk $\varphi(n, 6)$.

Za pomocą kół i tablicy rachunek, wyżej wykonany, robi się tak:

$$\varphi(100\,000, 4) = \varphi(100\,000, 6) - \varphi\left(\frac{100\,000}{p_7}, 6\right) - \varphi\left(\frac{100\,000}{p_8}, 7\right) -$$
$$- \ldots - \varphi\left(\frac{100\,000}{p_{14}}, 13\right)$$

$$\varphi(100\,000, 6) = \frac{100\,000}{30030} = 3.5760 + [\varphi(9910, 6) = \overset{6}{1901}] = 17280 +$$
$$+ \overset{6}{1901} = \mathbf{19181}$$

$$- \varphi\left(\frac{100\,000}{17}, 6\right) = \varphi(5882, 6) = \overset{6}{1128} \quad . \ . \ . \ . \ . \ . \ . \quad - \overset{6}{1128}$$

$$- \varphi\left(\frac{100\,000}{19}, 7\right) = \varphi(5263, 7) = \overset{6}{1009} - \overset{6}{59} = \overset{7}{950} \quad . \ . \quad - \overset{7}{950}$$

$$- \varphi\left(\frac{100\,000}{23}, 8\right) = \varphi(4347, 8) = \overset{6}{834} - (\overset{6}{49} + \overset{7}{43} = 92) = \overset{8}{742} \ . \quad - \overset{8}{742}$$

$$- \varphi\left(\frac{100\,000}{29}, 9\right) = \varphi(3448, 9) = \overset{6}{661} - (\overset{6}{41} + \overset{7}{36} - \overset{8}{28} = 105) \ . \quad - \overset{9}{556}$$

$$-\varphi\left(\frac{100\,000}{31}, 10\right) = \varphi(3225,10) = \overset{6}{6}18 - (\overset{6}{3}7 + \overset{7}{3}3 + \overset{8}{2}7 + \overset{9}{2}1 = 118) = \overset{10}{5}00 \ldots$$

$$\ldots - \overset{10}{5}00$$

$$-\varphi\left(\frac{100\,000}{37}, 11\right) = \varphi(2702,11) = \overset{6}{5}19 - (\overset{6}{3}2 + \overset{7}{2}8 + \overset{8}{2}3 + \overset{9}{1}6 + \overset{10}{1}4 = 113) =$$

$$= \overset{11}{4}06 \quad . \quad . \quad - \overset{11}{4}06$$

$$-\varphi\left(\frac{100\,000}{41}, 12\right) = \varphi(2439, 12) = \overset{6}{4}68 - (\overset{6}{2}9 + \overset{7}{2}5 + \overset{8}{2}0 + \overset{9}{1}5 + \overset{10}{1}2 + \overset{11}{8} =$$

$$= 109) = \overset{12}{3}59 \quad . \quad . \quad - \overset{12}{3}59$$

$$-\varphi\left(\frac{100\,000}{43}, 13\right) = \varphi(2325, 13) = \overset{6}{4}45 - (\overset{6}{2}7 + \overset{7}{2}4 + \overset{8}{1}9 + \overset{9}{1}4 + \overset{10}{1}2 + \overset{11}{8} + \overset{12}{5} =$$

$$= 109) = \overset{13}{3}36 \quad . \quad . \quad - \overset{13}{3}36$$

$$-4977$$

A zatem $\varphi(100\,000, 14) = \overset{6}{1}9181 - 4977 = \overset{14}{\mathbf{14204}}$;

W drugiej połowie formuły Meissela tylko człon -1 jest naturalnym, bo w numeracyi liczba *1* przez funkcyę $\varphi(m)$ pozostaje nietkniętą, a w wyrażeniu $\psi(m)$ nie powinna się znajdować.

Człon zaś: $-\sum_{s=1}^{s=\mu} \psi\left(\frac{n}{p_{m+s}}\right)$ od wartości $\varphi(n, m)$ odejmuje za wiele i dla tego staje się potrzebną restytucya przez człony $+ m(\mu + 1) + \frac{\mu(\mu-1)}{2}$. Jedyną racyą może być chyba potrzeba odróżnienia liczby liczb pierwszych, znajdujących się w pierwiastku sześciennym, $m = \psi\sqrt[3]{n}$, od reszty liczb pierwszych, znajdujących się w pierwiastku kwadratowym, gdyż $m + \mu = \psi\sqrt{n}$.

Liczby podzielne przez liczby pierwsze, wyrażone przez μ, z taką samą łatwością, jak przez sigma, obliczają się i przez funkcyę $\varphi(m)$, z tą tylko różnicą, że sigma bierze wartość $\psi\left(\frac{n}{p_k}\right)$ całą; funkcya zaś $\varphi(m)$ bierze tę samą wartość zmniejszoną $\varphi\left(\frac{n}{p_k}, k-1\right) = \psi\left(\frac{n}{p_k}\right) - (k-2)$ [1])

Można się o tem przekonać z następujących obliczeń; a najprzód sigma, i $\varphi(n, m+s)$.

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[1]) Wyrażenie $p_k = p_{m+s}$.

$p_{15} = 47; \psi\left(\frac{100\,000}{47}\right) = \psi(2127) = 319;$

$\varphi\left(\frac{100\,000}{47}, 14\right) = \varphi(2127), 14) = 319 - 13 = 306$

$p_{16} = 53; \psi\left(\frac{100\,000}{53}\right) = \psi(1886) = 289;$

$\varphi(1886, 15) = 289 - 14 = 275$

$p_{17} = 59; \psi\left(\frac{100\,000}{59}\right) = \psi(1694) = 264;$

$\varphi(1694, 16) = 264 - 15 = 249$

$p_{18} = 61; \psi\left(\frac{100\,000}{61}\right) = \psi(1639) = 259;$

$\varphi(1639, 17) = 259 - 16 = 243$

$p_{19} = 67; \psi\left(\frac{100\,000}{67}\right) = \psi(1492) = 237;$

$\varphi(1492, 18) = 237 - 17 = 220$

$p_{20} = 71; \psi\left(\frac{100\,000}{71}\right) = \psi(1408) = 222;$

$\varphi(1408, 19) = 222 - 18 = 204$

$p_{21} = 73; \psi\left(\frac{100\,000}{73}\right) = \psi(1369) = 219;$

$\varphi(1369, 20) = 219 - 19 = 200$

$p_{22} = 79; \psi\left(\frac{100\,000}{79}\right) = \psi(1268) = 205;$

$\varphi(1268, 21) = 205 - 20 = 185$

$p_{23} = 83; \psi\left(\frac{100\,000}{83}\right) = \psi(1204) = 197;$

$\varphi(1204, 22) = 197 - 21 = 176$

$p_{24} = 89; \psi\left(\frac{100\,000}{89}\right) = \psi(1123) = 188;$

$\varphi(1123, 23) = 188 - 22 = 166$

$p_{25} = 97; \psi\left(\frac{100\,000}{97}\right) = \psi(1030) = 172;$

$\varphi(1030, 24) = 172 - 23 = 149$

$$p_{26} = 101;\ \psi\left(\frac{100\,000}{101}\right) = \psi(990) = 166;$$

$$\varphi(990, 25) = 166 - 24 = 142$$

$$p_{27} = 103;\ \psi\left(\frac{100\,000}{103}\right) = \psi(970) = 163;$$

$$\varphi(970, 26) = 163 - 25 = 138$$

$$p_{28} = 107;\ \psi\left(\frac{100\,000}{107}\right) = \psi(934) = 158;$$

$$\varphi(934, 27) = 158 - 26 = 132$$

$$p_{29} = 109;\ \psi\left(\frac{100\,000}{109}\right) = \psi(917) = 156;$$

$$\varphi(917, 28) = 156 - 27 = 129$$

$$p_{30} = 113;\ \psi\left(\frac{100\,000}{113}\right) = \psi(884) = 153;$$

$$\varphi(884, 29) = 153 - 28 = 125$$

$$p_{31} = 127;\ \psi\left(\frac{100\,000}{127}\right) = \psi(787) = 138;$$

$$\varphi(787, 30) = 138 - 29 = 109$$

$$p_{32} = 131;\ \psi\left(\frac{100\,000}{131}\right) = \psi(763) = 135;$$

$$\varphi(763, 31) = 135 - 30 = 105$$

$$p_{33} = 137;\ \psi\left(\frac{100\,000}{137}\right) = \psi(729) = 129;$$

$$\varphi(729, 32) = 129 - 31 = 98$$

$$p_{34} = 139;\ \psi\left(\frac{100\,000}{139}\right) = \psi(719) = 128;$$

$$\varphi(719, 33) = 128 - 32 = 96$$

$$p_{35} = 149;\ \psi\left(\frac{100\,000}{149}\right) = \psi(671) = 121;$$

$$\varphi(671, 34) = 121 - 33 = 88$$

$$p_{36} = 151;\ \psi\left(\frac{100\,000}{151}\right) = \psi(662) = 121;$$

$$\varphi(662, 35) = 121 - 34 = 87$$

$$p_{37} = 157;\ \psi\left(\frac{100\,000}{157}\right) = \psi(636) = 115;$$

$$\varphi(636, 36) = 115 - 35 = 80$$

$$p_{38} = 163; \psi\left(\frac{100\,000}{163}\right) = \psi(613) = 112;$$

$$\varphi(613, 37) = 112 - 36 = 76$$

$$p_{39} = 167; \psi\left(\frac{100\,000}{167}\right) = \psi(598) = 108;$$

$$\varphi(598, 38) = 108 - 37 = 71$$

$$p_{40} = 173; \psi\left(\frac{100\,000}{173}\right) = \psi(578) = 106;$$

$$\varphi(578, 39) = 106 - 38 = 68$$

$$p_{41} = 179; \psi\left(\frac{100\,000}{179}\right) = \psi(558) = 102;$$

$$\varphi(558, 40) = 102 - 39 = 63$$

$$p_{42} = 181; \psi\left(\frac{100\,000}{181}\right) = \psi(552) = 101;$$

$$\varphi(552, 41) = 101 - 40 = 61$$

$$p_{43} = 191; \psi\left(\frac{100\,000}{191}\right) = \psi(523) = 99;$$

$$\varphi(523, 42) = 99 - 41 = 58$$

$$p_{44} = 193; \psi\left(\frac{100\,000}{193}\right) = \psi(518) = 97;$$

$$\varphi(518, 43) = 97 - 42 = 55$$

$$p_{45} = 197; \psi\left(\frac{100\,000}{197}\right) = \psi(507) = 96;$$

$$\varphi(507, 44) = 96 - 43 = 53$$

$$p_{46} = 199; \psi\left(\frac{100\,000}{199}\right) = \psi(502) = 95;$$

$$\varphi(502, 45) = 95 - 44 = 51$$

$$p_{47} = 211; \psi\left(\frac{100\,000}{211}\right) = \psi(473) = 91;$$

$$\varphi(473, 46) = 91 - 45 = 46$$

$$p_{48} = 223; \psi\left(\frac{100\,000}{223}\right) = \psi(448) = 86;$$

$$\varphi(448, 47) = 86 - 46 = 40$$

$$p_{49} = 227; \psi\left(\frac{100\,000}{227}\right) = \psi(440) = 85;$$

$$\varphi(440, 48) = 85 - 47 = 38$$

$$p_{50} = 229;\ \psi\left(\frac{100\,000}{229}\right) = \psi(436) = 84;$$

$$\varphi(436, 49) = 84 - 48 = 36$$

$$p_{51} = 233;\ \psi\left(\frac{100\,000}{233}\right) = \psi(429) = 82;$$

$$\varphi(429, 50) = 82 - 49 = 33$$

$$p_{52} = 239;\ \psi\left(\frac{100\,000}{239}\right) = \psi(418) = 80;$$

$$\varphi(418, 51 = 80 - 50 = 30$$

$$p_{53} = 241;\ \psi\left(\frac{100\,000}{241}\right) = \psi(414) = 80;$$

$$\varphi(414, 52) = 80 - 51 = 29$$

$$p_{54} = 251;\ \psi\left(\frac{100\,000}{251}\right) = \psi(398) = 78;$$

$$\varphi(398, 53) = 78 - 52 = 26$$

$$p_{55} = 257;\ \psi\left(\frac{100\,000}{257}\right) = \psi(389) = 77;$$

$$\varphi(389, 54) = 77 - 53 = 24$$

$$p_{56} = 263;\ \psi\left(\frac{100\,000}{263}\right) = \psi(380) = 75;$$

$$\varphi(380, 55) = 75 - 54 = 21$$

$$p_{57} = 269;\ \psi\left(\frac{100\,000}{269}\right) = \psi(371) = 73;$$

$$\varphi(371, 56) = 73 - 55 = 18$$

$$p_{58} = 271;\ \psi\left(\frac{100\,000}{271}\right) = \psi(369) = 73;$$

$$\varphi(369, 57) = 73 - 56 = 17$$

$$p_{59} = 277;\ \psi\left(\frac{100\,000}{277}\right) = \psi(361) = 72;$$

$$\varphi(361, 58) = 72 - 57 = 15$$

$$p_{60} = 281;\ \psi\left(\frac{100\,000}{281}\right) = \psi(355) = 71;$$

$$\varphi(355, 59) = 71 - 58 = 13$$

$$p_{61} = 283;\ \psi\left(\frac{100\,000}{283}\right) = \psi(353) = 71;$$

$$\varphi(353, 60) = 71 - 59 = 12$$

$$p_{62} = 293;\ \psi\left(\frac{100\,000}{293}\right) = \psi(341) = 68;$$

$$\varphi(341, 61) = 68 - 60 = 8$$

$$p_{63} = 307;\ \psi\left(\frac{100\,000}{307}\right) = \psi(325) = 66;$$

$$\varphi(325, 62) = 66 - 61 = 5$$

$$p_{64} = 311;\ \psi\left(\frac{100\,000}{311}\right) = \psi(321) = 66;$$

$$\varphi(321, 63) = 66 - 62 = 4$$

$$p_{65} = 313;\ \psi\left(\frac{100\,000}{313}\right) = \psi(319) = 66;$$

$$\varphi(319, 64) = 66 - 63 = 3$$

6614

4676

A zatem $-\sum_{s=1}^{s=51} \psi\left(\frac{100\,000}{p_{14+s}}\right) = -$ **6614**

$+ m(\mu + 1) = 14 . 52 =$ **728**

$+ \frac{\mu(\mu - 1)}{2} = \frac{51 . 50}{2} = \frac{2550}{2} =$ **1275**

$$\psi(100\,000) = \varphi(100\,000, 14) + 14 . 52 + \frac{51 . 50}{2} - 1 - \sum_{s=1}^{s=51} \psi\left(\frac{100\,000}{p_{14+s}}\right) =$$

$$= 14204 + 728 + 1275 - 1 - 6614 = 16207 - 6615 = \mathbf{9592};$$

Funkcya zaś $\varphi(m)$ bez sigmy i bez jej restytucyi daje liczbę liczb pierwszych, do której należy dodać usunięte przez nią [funkcyą $\varphi(m)$] liczb pierwszych 65, a odjąć 1; czyli $\psi(100\,000) = \varphi(100\,000, 14) - \varphi(100\,000, 65) + 65 - 1 = 14204 - 4676 - 1 + 65 = 14269 - 4677 =$ **9592.**

Stąd wnioskuję, że wzór $\psi(n) = \varphi(n, \psi\sqrt{n}) + \psi\sqrt{n} - 1$ jest prostszy i naturalniejszy od Meisselowskiego i nie trudniejszy do obliczenia.