11.

ON THE RELATION OF STURM'S AUXILIARY FUNCTIONS TO THE ROOTS OF AN ALGEBRAIC EQUATION.

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THE author availed himself of the present meeting of the British Association to bring under the more general notice of mathematicians his discovery, made in the year 1839, of the real nature and constitution of the auxiliary functions (so-called) which Sturm makes use of in *locating* the roots of an equation: these are obtained by proceeding with the left-hand side of the equation and its first differential coefficient as if it were our object to obtain their greatest common factor; the successive remainders, with their signs *alternately* changed and preserved, constitute the functions in question. Each of these may be put under the form of a fraction, the denominator of which is a perfect square, or in fact the product of *many*: likewise the numerator contains a huge heap of factors of a similar form.

These therefore, as well as the denominator, since they cannot influence the series of *signs*, may be rejected; and furthermore we may, if we please, again make every other function, beginning from the last but one, change its sign, if we consent to use changes wherever Sturm speaks of continuations of sign, and *vice versâ*.

The functions of Sturm, thus modified and purged of irrelevancy, the author, by way of distinction, and still to attribute honour where it is really most due, proposes to call "Sturm's Determinators"; and he proceeds to lay bare the internal anatomy of these remarkable forms.

He uses the Greek letter "ζ" to indicate that the squared product of the differences of the letters before which it is prefixed is to be taken.

Let the roots of the equation be called respectively $a, b, c, e \ldots l$, the determinators taken in the inverse order are as follows:—

$$\zeta(a, b, c, e \ldots l).$$

$$\Sigma\zeta(b, c, e \ldots l)\, x - \Sigma a\zeta(b, c, e \ldots l).$$

$$\Sigma\zeta(c, e \ldots l)\, x^2 - \Sigma(a+b) . \zeta(c, e \ldots l)\, x + \Sigma ab . \zeta(c, e \ldots l).$$

*   *   *   *   *   *   *

$$\Sigma\{\zeta(k, l)(x-a)(x-b)(x-c)(x-e) \ldots (x-h)\}.$$

It may be here remarked, that the work of assigning the total number of real and of imaginary roots falls exclusively upon the coefficients of the leading terms, which the author proposes to call "Sturm's Superiors": these superiors are only *partial* symmetric functions of the *squared differences*, but *complete* symmetric functions of the *roots themselves*, differing in the former respect from those other (at first sight similar-looking) functions of the squared differences of the roots, in which, from the time of Waring downwards, the conditions of reality have been sought for. It seems to have escaped observation, that the series of terms constituting any one of the coefficients in the equation of the squares of the differences (with the exception of the first and last) each admit of being separated and classified into various subordinate groups in such a way, that instead of being treated as a single symmetric function of the *roots*, they ought to be viewed as aggregates of many. In fact, Sturm's superior No. 1 is identical with Waring's coefficient No. 1; Sturm's superior No. 2 is a *part* of Waring's coefficient No. 3; Sturm's superior No. 3 is a *part* of Waring's coefficient No. 6; and so forth till we come to Sturm's final superior, which is again coextensive and identical with the last coefficient in the equation of the squares of the differences. The theory of symmetric functions of forms which are themselves symmetric functions of simple letters, or even of other forms, the author states his belief is here for the first time shadowed forth, but would be beside his present object to enter further into. He would conclude by calling attention to the importance to the general interests of algebraical and arithmetical science that a searching investigation should be instituted for showing, *à priori*, how, when a set of quantities is known to be made up partly of possible and partly of *pairs* of impossible values, symmetrical functions of these, one less in number than the quantities themselves, may be formed, from the signs of the ratios of which to unity and to one another the respective amounts of possible and impossible quantities may at once be inferred: in short, we ought not to rest satisfied, until, from the very *form* of Sturm's Determinators, without caring to know how they have been obtained, we are able to pronounce upon the uses to which they may be applied.