29.

ON THE INTERSECTIONS OF TWO CONICS.

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LET the two conics be written

$$U = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

$$V = \alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha' yz + 2\beta' zx + 2\gamma' xy = 0,$$

and make

$$U + \lambda V = Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy.$$

In my paper in the last number of the *Journal**, I showed that the case of intersection of the two conics in two points was distinguishable from all other cases by the equation $\square(U + \lambda V) = 0$ having two imaginary roots. When all the roots are real, the curves either intersect in four points or not at all.

On the former supposition,

$$-C'^2 + AB, \quad -A'^2 + BC, \quad -B'^2 + CA,$$

which are quadratic functions of λ, will be negative for all three values of λ. On the contrary supposition, one value of λ will make all these three quantities negative, but the other two values with each make them all three positive.

Hence we obtain a symmetrical criterion (which I strangely omitted to state in my former paper) by forming the quantity

$$A'^2 + B'^2 + C'^2 - AB - AC - BC.$$

A cubic equation

$$Ly^3 + My^2 + Ny + P = 0$$

may be then constructed, of which the three values of the above function corresponding to three values of λ will be the roots.

The condition for *real* intersection is that L, M, N, P should be all of the same sign. The conics being supposed real, L and P are necessarily in both cases of the same sign. The condition is therefore satisfied if either L, M,

[* p. 119 above.]

N, or M, N, P be of the same sign, and is consequently equivalent to the condition that $\frac{M}{L}$ and $\frac{N}{L}$ shall be both positive, or $\frac{N}{P}$ and $\frac{M}{P}$ both positive. It does not appear to be possible in the nature of the question to find a criterion for distinguishing between the two cases, dependent on the sign of one single function of the coefficients.

The case of double contact, abstraction being made of binary intersection, is a sort of intermediary state between intersection in four points and non-intersection; and accordingly, as shown in my former paper for this case, the two equal values of λ will make the three quantities

$$AB - C'^2, \quad BC - A'^2, \quad CA - B'^2$$

all real; so that two of the values of y corresponding to the equal values of λ are zero, and the criterion becomes nugatory as it ought to do.

Again, when the two conics do not intersect, I distinguished two cases according as they lie each without, or one within the other, that is, according as they have four common tangents or none.

But, as Mr Cayley has well remarked to me, a similar distinction exists when the conics intersect in four points; in that case also they may have four common tangents or not any: when they intersect in two points they have necessarily two and only two common tangents. There is no difficulty in separating these four cases.

Let the conics be written

$$(U) = \xi^2 + \eta^2 - \zeta^2,$$

$$(V) = A\xi^2 + B\eta^2 - C\zeta^2,$$

(U) and (V) being what U and V become when the coordinates are changed from x, y, z to ξ, η, ζ.

A, B, C are the three values of λ in the equation

$$\square (V - \lambda U) = 0.$$

If the curves intersect $A - C$, $B - C$ must have different signs, that is, C must be an intermediary quantity between A and B.

Again, the tangential equations to the conics expressed by the correlative system of coordinates will be

$$\xi_1^2 + \eta_1^2 - \zeta_1^2 = 0,$$

$$\frac{\xi_1^2}{A} + \frac{\eta_1^2}{B} - \frac{\zeta_1^2}{C} = 0;$$

and that these may have four real systems of roots,

$$\frac{1}{A} - \frac{1}{C}, \quad \frac{1}{C} - \frac{1}{B}$$

must have the same sign; and consequently, as $A - C$ and $C - B$ are supposed to have the same sign, A and B, and therefore all three A, B, C, have the same sign. We have therefore the following rule:

Let the equation in λ, namely, $\square(U+\lambda V)=0$, be called $\theta=0$, and the equation in y, above given, $\omega=0$. By an equation being congruent or incongruent, understand that its roots have all the same sign or not all the same sign.

Then ω congruent, θ congruent, implies that the intersections and common tangents are both real; ω congruent, θ incongruent, implies that the intersections are real, but the common tangents imaginary; ω incongruent, θ congruent, implies that the intersections and common tangents are both imaginary; ω incongruent, θ incongruent, implies that the intersections are imaginary, but the common tangents real.

In like manner, as the cases of contact of lines are limiting cases to those which relate to the relative configurations of their points of intersection, so the cases of contact of surfaces are limiting cases in which the characters which usually separate the different forms of their curve of intersection exist blended and indistinguishable. The first step therefore to the study of the particular species of the curve of the fourth degree*, in which two surfaces of the second degree intersect, is to obtain the analytical and geometrical characters of their various species of contact. Accordingly I have made an enumeration of these different species, no less than 12 in number, many of them highly curious and I believe unsuspected, which the reader may consult in the *Philosophical Magazine* for February, 1851†.

By the aid of these landmarks, I have little doubt, should time and leisure permit, of mapping out a natural arrangement of the principal distinctions of form between that class at least of lines in space of the fourth order which admit of being considered the complete intersection of two surfaces.

* I have found that the 16 points of spherical flexure in this curve are the four sets of four points in which it meets the four faces of the pyramid whose summits are the vertices of the four cones of the second degree in which the curve is completely contained, which 16 points reduce to 4 when the two surfaces have an ordinary contact, and to 1 when they have a cuspidal contact: of course in the case of contact the pyramid above described in a manner folds up and vanishes, as there are no longer 4 distinct vertices. I have found also that when the factors of $\square(U+\lambda V)$, (U and V being the characteristics of the two surfaces) are all unreal, the points of flexure are all unreal. When two factors are real and two imaginary, two of the faces of the pyramid (namely, its two real faces) will each contain one (and only one) pair of real points of flexure, and the other two planes none; and lastly, when the factors of $\square(U+\lambda V)$ are all real, then either all the points of flexure are imaginary, or else all the eight contained in a certain two of the pyramidal faces are real: and these two cases admit of being distinguished by a method analogous in its general features to that whereby I have shown in the text above how to distinguish between the cases of 4 real and 4 imaginary points of intersection of two conics. Where the two surfaces have an ordinary contact, the curve of intersection, it is well known, has a double point; and where the surfaces have a higher contact, the curve has a cusp. Thus in the fact of the 16 flexures reducing to 4 and to 1 in these respective cases, we see a beautiful analogy to what takes place with the 9 flexures of a plane curve of the third degree, which contract to 3 and 1, according as the curve has a double point or a cusp.

[† p. 219 below.]