32.

SKETCH OF A MEMOIR ON ELIMINATION, TRANSFORMATION, AND CANONICAL FORMS.

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THERE exists a peculiar system of analytical logic, founded upon the properties of zero, whereby, from dependencies of equations, transition may be made to the relations of functional forms, and *vice versâ*: this I call the logic of characteristics.

The resultant of a given system of homogeneous equations of as many variables, is the function whose nullity implies and is implied by the possibility of their coexistence, that is, is the characteristic of such possibility; but inasmuch as any numerical product of any power of a characteristic is itself an equivalent characteristic, in order to give definiteness to the notion of a resultant, it must further be restricted to signify the characteristic taken in the *lowest form* of which it *in general* admits.

The following very general and important proposition for the change of the independent variables in the process of elimination, is an immediate consequence of the doctrine of characteristics.

Let there be two sets of homogeneous forms of function;

the 1st, $\phi_1, \ \phi_2 \cdots \phi_n,$

the 2nd, $\psi_1, \ \psi_2 \cdots \psi_n.$

Let the results of applying these forms to any sets of n variables be called

$$(\phi_1), \ (\phi_2) \ldots (\phi_n),$$

$$(\psi_1), \ (\psi_2) \ldots (\psi_n);$$

then will the resultant (in respect to those variables) of

$$\phi_1 \{(\psi_1), \ (\psi_2) \ldots (\psi_n)\},$$

$$\phi_2 \{(\psi_1), \ (\psi_2) \ldots (\psi_n)\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\phi_n \{(\psi_1), \ (\psi_2) \ldots (\psi_n)\},$$

be the product of powers (assignable by the law of homogeneity) of the separate resultants of the two systems,

$$\{(\phi_1),\ (\phi_2)\ \dots\ (\phi_n)\},$$
$$\{(\psi_1),\ (\psi_2)\ \dots\ (\psi_n)\}.$$

By means of the doctrine of characteristics the following general problem may be resolved.

Given any number of functions of as many letters, and an inferior number of functions of the same inferior number of letters, obtained by combining, *inter se*, in a known manner, the given functions, to determine the factor by which, the resultant of the reduced system being divided, the resultant of the original system may be obtained.

If in the theorem for the change of the independent variables both sets of forms of functions be taken linear, we obtain the common rule for the multiplication of determinants: if we take one set linear and the other not, we deduce two rules, namely, That the resultant of a given set of functional forms of a given set of variables, enters as a factor into the resultant,

1st, of linear functions of the given functions of the given variables;

2nd, of the given functions of linear functions of the given variables:

the extraneous factor in each case being a power of what may be conveniently termed the *modulus of transformation*, that is, the resultant of the imported linear forms of functions.

From the second of these rules we obtain the law first stated I believe for functions beyond the second degree by Mr Boole, to wit, that the determinant of any homogeneous algebraical function (meaning thereby the resultant of its first partial differential coefficients) is unaltered by any linear transformations of the variables, except so far as regards the introduction of a power of the modulus of transformation. This is also abundantly apparent from the fact, that the nullity of such determinant implies an immutable, that is, a fixed and inherent, property of a certain corresponding geometrical locus.

There exist (as is now well known) other functions besides the determinant, called by their discoverer (Mr Cayley) hyperdeterminants, gifted with a similar property of immutability. I have discovered a process for finding hyperdeterminants of functions of any degree of any number of letters, by means of a process of Compound Permutation. All Mr Cayley's forms for functions of two letters may be obtained in this manner by the aid of one of the two processes (to wit, that one which will hereafter be called the derivational process), for passing from immutable constants to immutable forms. Such constants and forms, derived from given forms, may be best termed adjunctive; a term slightly varied from that employed by M. Hermite in a more restricted sense.

The two processes alluded to may be termed respectively appositional and derivational. The appositional is founded upon the properties of the binary function $x\xi + y\eta + z\zeta + \ldots$; in which, whether we substitute linear functions of x, y, z, &c., or linear functions of ξ, η, ζ, &c., in place of x, y, z, &c., or ξ, η, ζ, &c., the result is the same.

Consequently, if we apply the form ϕ to $\xi, \eta \ldots \zeta$, and take any constant (in respect to $\xi, \eta \ldots \zeta$) adjunctive to

$$\phi(\xi, \eta \ldots \zeta) + (x\xi + y\eta + \ldots + z\zeta + kt)\, t^{n-1},$$

calling this quantity $\psi(x, y \ldots z, t)$, the form ψ is evidently adjunctive to the form ϕ: and if we expand so as to obtain

$$\psi(x, y \ldots z, t) = \psi_1(x, y \ldots z)\, t^{\alpha} + \psi_2(x, y \ldots z)\, t^{\beta} + \text{\&c.},$$

it is evident ψ_1, ψ_2, &c. will be each separately adjunctive to ϕ. These forms, when ψ is obtained by finding the determinant in respect to $\xi, \eta \ldots \zeta$ of S, are, in fact, identical with Hermite's "formes adjointes."

The derivational mode of generating forms from constants depends upon the property of the operative symbol

$$\chi = \xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots + \zeta \frac{d}{dz},$$

applied to ϕ a function of $x, y \ldots z$; namely, that if in ϕ, in place of these letters, we write linear functions thereof, to wit $x', y' \ldots z'$, we may write

$$\chi = \xi' \frac{d}{dx'} + \eta' \frac{d}{dy'} + \ldots + \zeta' \frac{d}{dz'},$$

where $\xi', \eta' \ldots \zeta'$ will be the same functions of $\xi, \eta \ldots \zeta$ that $x', y' \ldots z'$ are of $x, y \ldots z$.

Suppose now, in the first place, that in regard to $\xi, \eta \ldots \zeta$, $\psi(x, y \ldots z)$ is adjunctive to $\chi^r \phi(x, y \ldots z)$; then is the form ψ adjunctive to the form ϕ, for on changing $x, y \ldots z$ to $x', y' \ldots z'$,

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots + \zeta \frac{d}{dz}\right)^r \phi(x', y' \ldots z')$$

becomes
$$\left(\xi' \frac{d}{dx'} + \eta' \frac{d}{dy'} + \ldots + \zeta' \frac{d}{dz'}\right) \phi(x', y' \ldots z');$$

and consequently $\psi(x, y \ldots z)$ becomes $\psi(x', y' \ldots z')$, multiplied by a power of the modulus of transformation, the modulus of that transformation, be it well observed, whereby $x', y' \ldots z'$ would be replaced by $x, y \ldots z$, and not as in the appositional mode of that converse transformation according to which

$x, y \dots z$ would be replaced by $x', y' \dots z'$. It is on account of this converseness of the modes of transformation that the appositional and derivational modes of generating forms cannot except for a certain class of *restricted* linear transformations be combined in a single process. More generally, if instead of a single function $\chi^r \phi (x, y \dots z)$, we take as many such with different indices to χ as there are variables, and form either the resultant in respect to $\xi, \eta \dots \zeta$, or any other immutable constant in regard to those variables, (presuming in extension of the hyperdeterminant theory and as no doubt is the case, that such exist), every such resultant or other constant will give a form of function of $x, y \dots z$ adjunctive to the given form ϕ.

It may be shown that every such resultant so formed will contain ϕ as a factor.

Again, in the former more available determinant mode of generation, if we take the determinant in respect to $\xi, \eta \dots \zeta$, it may be shown that all the adjunctive functions so obtained will be algebraical derivees of the partial differential coefficients of ϕ in respect to $x, y \dots z$; that is to say, if these be respectively zero, all such adjunctive functions so derived, as last aforesaid, will be zero, or in other words, each such adjunctive is a syzygetic function of the partial differential coefficients of the primitive function.

To Mr Boole is due the high praise of discovering and announcing, under a somewhat different and more qualified form and mode of statement, this marvel-working process of derivational generation of adjunctive forms. I was led back to it, in ignorance of what Mr Boole had done, by the necessity which I felt to exist of combining Hesse's so-called functional determinant, under a common point of view with the common constant determinant of a function; under pressure of which sense of necessity, it was not long before I perceived that they formed the two ends of a chain of which Hesse's end exists for all homogeneous functions, but the other only when such functions are algebraical.

In fact, if we give to r every value from 2 upwards, the successive determinants in respect to $\xi, \eta \dots \zeta$ of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \dots + \zeta \frac{d}{dz}\right)^r \phi (x, y, z),$$

will produce the chain in question, which, when ϕ is algebraical and of n dimensions, comes to a natural termination when $r = n - 1$. The last member of and the number of terms in this chain are identical with the last member of and the number of terms in Sturm's auxiliary functions, when the variables are reduced to two. There is some reason to anticipate that this chain of functions may be made available in superseding Sturm's chain of auxiliaries; and if so, then the fatal hindrance to progress, arising from the unsymmetrical nature of the latter, is overcome, and we shall be able to pass from Sturm's theorem, which relates to the theory of Keno-themes, or Point-systems, to certain corresponding but much higher theories for lines, surfaces, and n-themes generally.

The restriction of space allowed to me in the present number of the *Journal* will permit me only to allude in the briefest terms to the theory of Relative Determinants, which, as it will be seen, plays an important part in the effectuation of the reductions of the higher algebraical functions to their simplest forms. Nor can the effect of the processes to be indicated be correctly appreciated without a knowledge of the circumstances under which the resultant of a *given* system of equations can sink in degree below the resultant of the *general* type of such system. Abstracting from the case when the equations separately, or in combination, subdivide into factors, this lowering of degree, as may be shown by the doctrine of characteristics, can only happen in one of two ways. Either the particular resultant obtained is a rational root of the general resultant, or the general resultant becomes zero for the case supposed, and the particular resultant is of a distinct character from the general resultant, being in fact the characteristic of the possibility not of the given system of equations being merely able to coexist (for that is already supposed), but of their being able to coexist for a certain system of values *other than* a given system or given systems. Such a resultant may be termed a Sub-resultant; the lowest resultant in the former case may be termed a Reduced-resultant. The theory of Sub-resultants is one altogether remaining to be constructed, and is well worthy equally of the attention of geometers and of analysts.

As to the theory of Relative Determinants, the object of this theory is to obtain the determinant resulting from eliminating as many variables as can be eliminated, chosen at pleasure from a set of variables greater in number than the equations containing them; and the mode of effecting this object is through the method of the indeterminate multiplier. To avoid the discussion of the theory of sub-resultants and other particularities, I shall content myself with giving the rule applicable to the case (the only one of which as yet a practical application has offered itself to me in the course of my present inquiries) when all but one of the functions are linear.

If U, L_1, $L_2 \ldots L_m$ be the first an n^c and the others linear functions of n variables, and it be desired to find the determinant of the resultant arising from the elimination of any m out of the n variables, the following is the rule:

Find the determinant, that is, the resultant of the partial differential coefficients in respect to the given variables, and of $\lambda_1, \lambda_2 \ldots \lambda_m$ of

$$U + L_1\lambda_1 + L_2\lambda_2 + \ldots + L_m\lambda_m.$$

This resultant, in its lowest form, will be always a rational $(n-1)$th root of the resultant of the homogeneous system of equations to which the system above given can be referred as its type; and this reduced resultant divided by a power (determinable by the law of homogeneity) of the resultant of $L_1, L_2 \dots L_m$, when all but the selected variables are made zero, will be the resultant determinant required*. As regards what has been said concerning the reducibility of the general typical resultant in the case before us, this is a consequence of, and may be brought into connexion with, the following theorem, which is easily demonstrable by the theory of characteristics. If $Q_1, Q_2 \dots Q_m$ be m homogeneous functions of m variables of the same degree, r of which enter in each equation only as simple powers uncombined with any of the other variables, then the degree of the reduced resultant is equal to the number of the equations multiplied by the $(m-r-1)$th power of the number of units in the degree of each, subject to the obvious exception that when r is m, (there being in fact but *one* step from $r=m-2$ to $r=m$), instead of r, $(r-1)$ must be employed in the above formula. As an example of a sub-resultant as distinguished from a reduced-resultant, I instance the case of three quadratics U, V, W, functions of x, y, z, in each of which no squared power of z is supposed to enter: it may easily be shown by my dialytic method that instead of six equations, between which to eliminate x^2, y^2, z^2, xy, xz, yz, we shall have only 5, the three original ones and two instead of three auxiliaries between which to eliminate x^2, y^2, xy, xz, yz, the *apparent* resultant is accordingly of the 9th instead of the 12th degree. But this is not the true characteristic of the possibility of the coexistence of the given systems, which in fact is zero, as is evidenced by the fact that they always *do* coexist, since they are always satisfiable by only *two* relations between the variables, to wit $x=0$, $y=0$. The apparent resultant is then something different, and what has been termed by the above a Sub-resultant.

I take this opportunity of entering my simple protest against the appropriation of my method of finding the resultant of any set of three equations of degrees equal or differing only by a unit, one from those of the other two, by Dr Hesse, so far as regards quadratic functions, without acknowledgment, four years after the publication of my memoir in the *Philosophical Magazine*: the fundamental idea of Dr Hesse's partial method is identical with that of my general one. Still more unjustifiable is the subsequent use of the *dialytic* principle, by the same author, equally without acknowledgment, and in cases where there is no peculiarity of form of procedure to give even a plausible ground for evading such acknowledgment. It is capable of moral proof that

* The same method applies not only to the Final or Constant Determinant, but likewise to all the Functional Determinants in the chain above described, extending upwards from this to the Hessian, or as it ought to be termed, the first Boolian Determinant.

what I had written on the matter was sufficiently known in Berlin and at Königsberg, at each epoch of Dr Hesse's use of the method.

I now proceed to the consideration of the more peculiar branch of my inquiry, which is as to the mode of reducing Algebraical Functions to their simplest and most symmetrical, or as my admirable friend M. Hermite well proposes to call them, their Canonical forms. Every quadratic function of any number of variables may always be linearly transformed into any other quadratic functions of the same, and that too in an infinite variety of ways; but in every other instance there will be only a limited number of ways, whereby, when possible, one form will admit of being transmuted into any other: and with the sole exception of a cubic function of two letters, such transmutation will never be possible, unless a certain condition, or certain conditions, be satisfied between the constants of the forms proposed for transmutation. The number of such conditions is the number of parameters entering into the canonical form, and is of course equal to the number of terms in the general form of the function diminished by the square of the number of letters. Thus there is one parameter in the canonical form for the biquadratic function of two and the cubic function of three letters, and no parameter in the cubic function of two letters. Hitherto no canonical forms have been studied beyond the cases above cited, but I have succeeded, as will presently be shown, in obtaining methods for reducing to their canonical forms functions with *two* and *four* parameters respectively. Owing to what has been remarked above, the theory of quadratic functions is a theory apart. Simultaneous transformation gives definiteness to that theory, but has no existence for any useful purpose for functions of the higher degrees. Where the theory of simultaneous transformation ends, that of canonical forms properly begins; and in what follows, the case of quadratic forms is to be understood as entirely excluded. Such exclusion being understood, there is no difficulty in assigning the canonical, that is, the simplest and most symmetrical general, form to which every function of two letters admits of being reduced by linear transformations. If the degree be odd, say $2m+1$, the canonical form will be

$$u_1^{2m+1} + u_2^{2m+1} + \dots + u_{m+1}^{2m+1};$$

if the degree be even, say $2m$, the canonical form will be

$$u_1^{2m} + u_2^{2m} + \dots + u_m^{2m} + K(u_1 u_2 \dots u_m)^2,$$

all the u's being linear functions of the two given variables. It is easy to extend an analogous mode of representation to functions of any number of letters. From the above we see that for cubic, biquadratic, and quintic functions of two letters, the canonical forms will be respectively

$$u^3 + v^3, \quad u^4 + v^4 + Ku^2v^2, \quad u^5 + v^5 + w^5,$$

with a linear relation in the last-named case between u, v, w.

First as to the reduction of any 4^c function to Cayley's form

$$u^4 + v^4 + Ku^2v^2.$$

This may be effected in a great variety of ways, of which the following is not the simplest as regards the calculations required, but the most obvious. Let the modulus of transformation, whereby the given biquadratic function, say $F(x, y)$, becomes transmuted into its canonical form, be called M; let the determinant of F be called D_1, and the determinant of the determinant in respect to ξ and η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^2 F(x, y),$$

which latter, for brevity's sake, may be termed the Hessian of F, (although in stricter justice the Boolian would be the more proper designation), be called D_2. Then, by examining the canonical form itself (which is as it were the very *palpitating heart* of the function laid bare to inspection), we shall obtain without difficulty the two equations

$$(1 - 9m^2)^2 = M^{12} D_1 \frac{1}{4^6},$$

$$m^2 (1 - 9m^2)^2 (m^2 - 1)^2 = M^{24} D_2 \frac{1}{12^{12} 4^4}.$$

Eliminating the unknown quantity M, we obtain

$$\frac{m^2 (m^2 - 1)^2}{(1 - 9m^2)^2} = c, \quad \text{or} \quad \frac{m^3 - m}{1 - 9m^2} = c^{\frac{1}{2}},$$

where c is a known quantity.

This *cubic* equation for finding m is of a peculiar form; it being easy to show *à priori*, by going back to the canonical form, that its three roots are m, $\theta(m)$, $\theta^2(m)$, where

$$\theta(m) = \frac{m - 1}{3m + 1},$$

θ being a periodical form of function such that $\theta^3(m) = m$.

This it is which accounts for the simple expression for m, that may be obtained by solving the cubic above given. A better practical mode is to take, instead of the determinant of the given function and its Hessian, the two hyperdeterminants and eliminate as before: a cubic equation having precisely the same properties, and in fact virtually identical with the former, will result. When m and consequently M are found, there is no difficulty whatever, calling the given function F and its Hessian $H(F)$, in forming linear functions of the two, as

$$\left.\begin{array}{l} \phi(m) F + \psi(m) H(F) \\ \phi_1(m) F + \psi_1(m) H(F) \end{array}\right\},$$

which shall be equal to, that is, identical with, $(u^2 + v^2)^2$ and u^2v^2, whence u and v are completely determined.

Another and interesting mode of solution is to take, besides the given function F and its Hessian, either the *second* Hessian or the post-Hessian of the given function, by the post-Hessian understanding the determinant in respect of ξ and η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^3 F:$$

any three of the four functions will be linearly related, and it may be shown that, calling either the second Hessian (that is, the Hessian of the Hessian) or the post-Hessian H', we shall have

$$H'(F) + aH(F) + bF = 0,$$

where a and b will be *rational* and *integer* functions of the coefficients of F, and numerical multiples of two quantities R and S, such that the determinant of F will be equal $R^3 + S^2$; and this, be it observed, without any previous knowledge of the existence of these hyperdeterminants R and S.

If now we go to Hesse's form for a cubic function of three letters, we shall find that precisely similar modes of investigation apply step for step. Calling the function F and its Hessian $H(F)$, and the post-Hessian or second Hessian at choice $H'(F)$, we shall find

$$H'(F) + mSH(F) + nR^2F = 0,$$

where m and n are numerical quantities and $R^3 + S^2$ equal the determinant of F. It is interesting to contrast this equation with the one previously mentioned as applicable to the 4[c] functions of two letters, namely,

$$H'(F) + mRH(F) + nSF = 0.$$

In both instances there is no difficulty in assigning the relations between the original R and S, and the R and S of any adjunctive form. All Aronhold's results may be thus obtained and further extended without the slightest difficulty. As regards the equation for finding the parameter in Hesse's canonical form for the cubic of three letters, this will be of the 4th degree in respect to the cube of the parameter, and the roots will be functionally representable as

$$x; \quad \theta(x); \quad \phi(x); \quad \psi(x),$$

where

$$\theta^2(x) = \phi^2(x) = \psi^2(x) = x;$$
$$\theta\phi(x) = \phi\theta(x) = \psi(x),$$
$$\phi\psi(x) = \psi\phi(x) = \theta(x),$$
$$\psi\theta(x) = \theta\psi(x) = \phi(x);$$

owing to which property the equation is soluble under the peculiar form observed by Aronhold.

I pass on now to a brief account of the method, or rather of a method (for I doubt not of being able to discover others more practical), of reducing a function of the 5th degree of two letters (say of x and y) to its canonical form $u^5+v^5+w^5$, subject to the linear relation $au+bv+cw=0$, where the ratios $a:b:c$, and the linear relations between u, v, w and the two given variables are the objects of research. Here I have found great aid from the method of Relative Determinants; and I may notice that the successful application of more compendious methods to the question would be greatly facilitated were there in existence a theory of Relative Hyperdeterminants, which is still all to form, but which I little doubt, with the blessing of God, to be able to accomplish. It may some little facilitate the comprehension of what follows, if c be considered as representing unity.

Calling as before the given quintic function F, the modulus of transformation M, the Hessian and post-Hessian of F, H and H', and its ordinary or constant determinant D, we shall find

$$a^2v^3w^3+b^2w^3u^3+c^2u^3v^3=M^2H,$$

and

$$P_1P_2P_3P_4=M^6H',$$

where

$$P_1=\ \ a^{\frac{3}{2}}vw+b^{\frac{3}{2}}wu+c^{\frac{3}{2}}uv,$$

$$P_2=\ \ a^{\frac{3}{2}}vw-b^{\frac{3}{2}}wu-c^{\frac{3}{2}}uv,$$

$$P_3=-a^{\frac{3}{2}}vw+b^{\frac{3}{2}}wu-c^{\frac{3}{2}}uv,$$

$$P_4=-a^{\frac{3}{2}}vw-b^{\frac{3}{2}}wu+c^{\frac{3}{2}}uv;$$

also $D=M^{20}$ multiplied by the product of the sixteen values of

$$a^{\frac{5}{4}}+b^{\frac{5}{4}}(1)^{\frac{1}{4}}+c^{\frac{5}{4}}(1)^{\frac{1}{4}}.$$

From the above equations it may be shown that H' (a known function of the 8th degree of the given variables x, y) must be capable of being thrown under the form

$$L\{(x-a_1y)(x-a_2y)\times(x-a_3y)(x-a_4y)\\ \times(x-a_5y)(x-a_6y)\times(x-a_7y)(x-a_8y)\},$$

where

$$(a_1-a_2)^2\times(a_3-a_4)^2\times(a_5-a_6)^2\times(a_7-a_8)^2$$

$$=\frac{D}{L^2}=K,$$

so that K is a known quantity*. Accordingly the said equation of the 8th degree, considered as an algebraical equation in $\frac{x}{y}$, may by known methods be

* Or in other words, the post-Hessian determinant of a given function in two letters of the second degree, may be divided into four quadratic factors in such a way that the product of the determinants of these several factors shall be equal to the determinant of the given function.

found by means of equations not exceeding the 4th or even the 3rd degree: in fact, to do this it is only necessary to form the equation to the squares of the differences of the roots of $\frac{x}{y}$ in the equation $H' \div y^8 = 0$, which new equation will be of the 28th degree. If we then form two other equations of the 378th degree, one having its roots equal to $\sqrt{K}$ multiplied by the binary products of the twenty-eight roots of the equation last named, the other to $\sqrt{K}$ multiplied by the reciprocal of such binary products, the left-hand members of these two equations expressed under the usual form will have a factor in common, which may be found by the process of common measure and will be of the 6th degree, whose roots consisting of three pairs of reciprocals may be found by the solution of cubics only.

In this way, by means of cubics and quadratics,

$$(a_1 - a_2)^2, \quad (a_3 - a_4)^2, \quad (a_5 - a_6)^2, \quad (a_7 - a_8)^2,$$

can be found, which being known,

$$a_1 a_2, \quad a_3 a_4, \quad a_5 a_6, \quad a_7 a_8,$$

can be determined in pairs by means of quadratics from the equation $H' \div y^8 = 0$. This being supposed to be done, we have

$$P_1 = fL_1,$$
$$P_2 = gL_2,$$
$$P_3 = hL_3,$$
$$P_4 = kL_4,$$

where L_1, L_2, L_3, L_4, are *known* quadratic functions of x and y. To determine the ratios of f, g, h, k, we have three equations* obtained from the identity

$$fL_1 + gL_2 + hL_3 + kL_4 (= P_1 + P_2 + P_3 + P_4) = 0;$$

$f : g : h : k$ being known, $fL_1 : gL_2 : hL_3 : kL_4$ are known ratios.

But
$$P_1 + P_2 = 2a^{\frac{3}{2}} vw,$$
$$P_1 + P_3 = 2b^{\frac{3}{2}} wu,$$
$$P_1 + P_4 = 2c^{\frac{3}{2}} uv.$$

Hence
$$a^{\frac{3}{2}} vw = \lambda P,$$
$$b^{\frac{3}{2}} wu = \lambda Q,$$
$$c^{\frac{3}{2}} uv = \lambda R,$$

where P, Q, R are known quadratic functions of x, y.

* For we must have the coefficients of x^2, xy and y^2 in

$$fL_1 + gL_2 + hL_3 + kL_4,$$

of all them zero.

Hence $a:b:c$ may be found by means of the identical equation

$$a^2w^3v^3 + b^2u^3w^3 + c^2v^3u^3 = H(F),$$

whereby the ratios $a^{-\frac{5}{2}} : b^{-\frac{5}{2}} : c^{-\frac{5}{2}}$ can be obtained without any further extraction of roots, showing that there is but one single true system of ratios $a^5 : b^5 : c^5$ applicable to the problem; $a:b:c$ being thus found, λ is easily determined, and thus finally u, v, w are found in terms of x and y*.

I have little doubt that a more expeditious mode of solution than the foregoing† will be afforded by an examination of the properties and relations of the *quadratic and cubic forms*, adjunctive to the general quintic functions, and indeed to every $(4n+1)^c$ function of two letters hereinbefore adverted to.

Sufficient space does not remain for detailing the steps whereby the general cubic function of *four* letters may, by aid of equations *not transcending the fifth degree*, be reduced to its canonical form $u^3 + v^3 + w^3 + p^3 + q^3$, wherein u, v, w, p, q are connected by a linear equation

$$au + bv + cw + dp + eq = 0;$$

the four ratios of whose coefficients $a:b:c:d:e$ give the necessary number $\frac{4.5.6}{1.2.3} - 4^2$ parameters furnished by the general rule. Suffice it for the present to say, that the analytical mode of solution depends upon a circumstance capable of the following geometrical statement: Every surface of the 4th degree represented by a function which is the Hessian to any given cubic function whatever of four letters, has lying upon it ten straight lines meeting three and three in ten points, and these ten points are the only points which enjoy the following property in respect to the surface of the 3rd degree denoted by equating to zero the cubical function in question, to wit, that the cone drawn from any one of them as vertex to envelop the surface, will meet it not in a continuous double curve of the 6th degree, but in two curves each of the 3rd degree, lying in *planes* which intersect in the ten lines respectively above named; so that to each of the ten points corresponds one of the ten lines: these ten points and lines are the intersections taken respectively three with three, and two with two, of *a single and unique system* of five principal planes appurtenant to every surface of the 3rd degree, and these planes are no other than those denoted by

$$u=0, \quad v=0, \quad w=0, \quad p=0, \quad q=0.$$

* The problem thus solved may be stated as consisting in reducing the general function $ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5$ to the form

$$(lx+my)^5 + (l'x+m'y)^5 + (l''x+m''y)^5.$$

† The coefficients in the reducing recurrent equation of the 6th degree in the process above detailed may rise to be of 541632 dimensions in respect to the original coefficients in F.

I have found also by the theory of Sub-resultants, that the analogy between lines and surfaces of the third degree, in regard to the existence of double and conical points, is preserved in this wise: that in the same way as a double point on a curve of the 3rd degree commands the existence of a double point on its Hessian, so does a conical point in a surface of the 3rd degree command over and above the 10 necessary, and so to speak natural conical points, at least one extra, that is to say an 11th conical point on *its* Hessian. And here for the present I must quit my brief and imperfect notice of this subject, composed amidst the interruptions and distractions of an official and professional life.

Observation. It may be somewhat interesting and instructive to my readers, to have a table of the successive scalar* determinants of a quintic function of two letters presented to them at a single glance. Preserving the notation above [page 193], we have the following expressions:

The given function $= u^5 + v^5 + w^5$,

its Hessian $= M^2 (a^2v^3w^3 + b^2w^3u^3 + c^2u^3v^3)$,

its post-Hessian $= M^6 \times$ the product of the *four* forms of

$$a^{\frac{3}{2}} vw + b^{\frac{3}{2}} (1)^{\frac{1}{2}} wu + c^{\frac{3}{2}} (1)^{\frac{1}{2}} uv;$$

its præter-post-Hessian $= M^{12} \times$ the product of the *nine* forms of

$$a^{\frac{4}{3}} v^{\frac{1}{3}} w^{\frac{1}{3}} + b^{\frac{4}{3}} (1)^{\frac{1}{3}} w^{\frac{1}{3}} u^{\frac{1}{3}} + c^{\frac{4}{3}} (1)^{\frac{1}{3}} u^{\frac{1}{3}} v^{\frac{1}{3}},$$

and the final determinant $= M^{20} \times$ the product of the *sixteen* forms of

$$a^{\frac{5}{4}} + (1)^{\frac{1}{4}} b^{\frac{5}{4}} + (1)^{\frac{1}{4}} c^{\frac{5}{4}}.$$

The success of the method applied depends (as above shown) upon the fact of a certain function of the roots of the post-Hessian (which is an octavic function of the variables) being known, which fact *hinges* upon the circumstance that

$$(M^6)^2 \times (M^2)^4 = M^{20}.$$

P.S. I have much pleasure in subjoining the cubical hyperdeterminant of the 12th degree function of two letters, worked out upon the principle of Compound Permutation hinted at in the foregoing pages, for which I am indebted to the kindness and skill of my friend Mr Spottiswoode.

* By which I mean the determinants in respect to ξ, η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^r F(xy).$$

The function being called

$$ax^{12} + 12bx^{11}y + \frac{12.11}{2}cx^{10}y^2 + \&c. \dots + ly^{12},$$

the following is* its cubical hyperdeterminant:

$$\begin{aligned} &agm - 6ahl + 15aik + 10aj^2 - 6bfm, \\ &- 24bhk + 30bgl + 20bij - 24cfl + 114cgk, \\ &- 145ci^2 + 50chj + 15cem + 20cgi + 20ch^2, \\ &- 400dgj + 280dhi + 20del + 50dfe + 10d^2k, \\ &+ 385egi - 135e^2k - 290eh^2 + 705fgh, \\ &- 330f^2i - 50g^3. \end{aligned}$$

Mr Spottiswoode will I hope publish the work itself in the next number of the *Journal*, in which I shall also show how the hyperdeterminants of the cubical function of three letters, Aronhold's S and T, may be similarly obtained.

[* See below, p. 202.]