33.

ON THE GENERAL THEORY OF ASSOCIATED ALGEBRAICAL FORMS.

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THE following brief exposition of the general theory of Associated Forms, as far as it has been as yet developed by the labours or genius of mathematicians, is intended as elucidatory and, to a certain extent, emendative of some of the statements in my paper* on Linear Transformations, in the preceding number of the *Journal*.

In the first place, let a linear equivalent of any given homogeneous function be understood to mean what the function becomes when linear functions of the variables are substituted in place of the variables themselves, subject to the condition of the modulus of transformation (that is, the value of the determinant formed by the coefficients of transformation) being unity.

Secondly, let two square arrays of terms (the determinants corresponding to each of which are unity) be said to be complementary when each term in the one square is equal to the value of what the determinant represented by the other square becomes when the corresponding term itself is taken unity, but all the other terms in the same line and column with it are taken zero. This relation between the two squares is well known to be reciprocal. Thus, for instance,

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}$$

will be said to be reciprocally complementary to one another when the two determinants which they represent are each unity, and when we have

[* p. 184, above.]

$$a = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \beta' & \gamma' \\ 0 & \beta'' & \gamma'' \end{vmatrix} \qquad \alpha = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b' & c' \\ 0 & b'' & c'' \end{vmatrix}$$

$$b = \begin{vmatrix} 0 & 1 & 0 \\ \alpha' & 0 & \gamma' \\ \alpha'' & 0 & \gamma'' \end{vmatrix} \qquad \beta = \begin{vmatrix} 0 & 1 & 0 \\ a' & 0 & c' \\ a'' & 0 & c'' \end{vmatrix}$$

$$b' = \begin{vmatrix} \alpha & 0 & \gamma \\ 0 & 1 & 0 \\ \alpha'' & 0 & \gamma'' \end{vmatrix} \qquad \beta' = \begin{vmatrix} a & 0 & c \\ 0 & 1 & 0 \\ a'' & 0 & c'' \end{vmatrix}$$

&c. &c.

Accordingly, two transformations, say of $F(x, y, z)$ and $G(u, v, w)$ respectively, may be said to be concurrent when in F for x, y, z, we write

$$ax + by + cz,$$
$$a'x + b'y + c'z,$$
$$a''x + b''y + c''z;$$

and in G for u, v, w, we write

$$au + bv + cw,$$
$$a'u + b'v + c'w,$$
$$a''u + b''v + c''w;$$

but complementary when for u, v, w, we write

$$\alpha u + \beta v + \gamma w,$$
$$\alpha' u + \beta' v + \gamma' w,$$
$$\alpha'' u + \beta'' v + \gamma'' w;$$

a, b, c, &c., α, β, γ, &c. being related in the manner antecedently explained.

Two forms, each of the same number of variables, may be said to be associate forms when the coefficients of the one are functions of those of the other; and when it happens that the coefficients of the first are all explicit functions of those of the second, the latter may be termed the originant and the former the derivant.

If now all the linear equivalents of one or of two associated forms are similarly related to corresponding linear equivalents of the other, so that each may be derived from each by the same law, the forms so associated will be said to be concomitant each to the other. This concomitance may be of two kinds, and very probably, in the nature of things, only of the two kinds about to be described.

The first species of concomitance is defined by the corresponding equivalents of the two associated forms being deduced by precisely similar, or, as we have expressed it, concurrent transformations or substitutions, each from its given primitive. The second species of concomitance is defined by the corresponding equivalents being deduced not by similar but by contrary, that is, reciprocal or complementary substitutions. Concomitants of the first kind may be called covariants; concomitants of the second kind may be called contravariants. When of the two associated forms one is a constant, the distinction between co- and contra-variants disappears, and the constant may be termed an invariant of the form with which it is associated*. It follows readily from these definitions that a covariant of a covariant and a contravariant of a contravariant are each of them covariants; but a covariant of a contravariant and a contravariant of a covariant are each of them contravariants; and also that an invariant, whether of a covariant or of a contravariant, is an invariant of the original function†.

It will also readily be seen that as regards functions of two letters a contravariant becomes a covariant by the simple interchange of x, y with $-y$, x, respectively. Covariants are Mr Cayley's hyperdeterminants; contravariants include, but are not coincident with, M. Hermite's formes-adjointes, if we understand by the last-named term such forms as may be derived by the process described by M. Hermite in the third of his letters to M. Jacobi, "Sur différents objets de la Théorie des Nombres," (which process is an extension of that employed for determining the polar reciprocal of an algebraical locus‡). M. Hermite appears, however, elsewhere to have used

* Accordingly an invariant to a given form may be defined to be such a function of the coefficients of the form, as remains absolutely unaltered when instead of the given form any linear equivalent thereto is substituted. Of course if the determinant of the coefficients of the transformations correspondent to the respective equivalents be not taken unity as supposed in this definition, the effect will be merely to introduce as a multiplier some power of the determinant formed by the coefficients of transformation.

† It may likewise be shown that linear equivalents of covariants and contravariants are themselves related to one another as covariants and contravariants respectively, the transformations by which the equivalents are obtained being taken concurrent in the one case and contrary or reciprocal in the other; and of course any algebraic function of any number of covariants is a covariant and of contravariants a contravariant.

‡ This has been further generalized by me in the theorem§ given in the last number of this *Journal*, where I have shown in effect that any invariant in respect to $\xi, \eta \ldots \theta$ of

$$f(\xi, \eta \ldots \theta) + (x\xi + y\eta + \ldots + t\theta + \rho)\rho^{n-1},$$

(f being supposed to be of the degree n) is a contravariant of $f(x, y \ldots t)$. When this invariant is the determinant of f, it may be shown that we obtain M. Hermite's theorem. It is somewhat remarkable that contravariants should have been in use among mathematicians as well in geometry as the theory of numbers (although their character as such was not recognized) before covariants had ever made their appearance. Invariants of course first came up with the theory of the equation to the squares of the differences of the roots of equations, the last term in such equation being an invariant. I believe that I am correct in saying that covariants first made their appearance in one of Mr Boole's papers, in this *Journal*; but Hesse's brilliant application

[§ p. 186 above.]

the term forme-adjointe in a sense as wide as that in which I employ contravariants. For instance, he has given a most remarkable theorem, which admits of being stated as follows:

If we have a function of any number of letters, say of x, y, z, as

$$ax^m + mbx^{m-1}y + mcx^{m-1}z + \frac{m(m-1)}{2} dx^{m-2}y^2 + \&c.,$$

and if I be any invariant of this function, then will

$$\left(x^m \frac{d}{da} + x^{m-1}y \frac{d}{db} + x^{m-1}z \frac{d}{dc} + x^{m-2}y^2 \frac{d}{dd} \ \&c.\right)^r I$$

be a "*forme-adjointe*" of the given function. It is perfectly true and admits of being very easily proved, as I shall show in your next number, that this is a contravariant of the given function*; but it is not (as far as I can see) a forme-adjointe in the sense in which the use of that word is restricted in the letter alluded to. If, however, we adopt as the *definition* of *formes-adjointes* generally, that property in regard to their transformées which M. Hermite has demonstrated of the particular class treated of by him in the letter alluded to, then his formes-adjointes become coincident with my contravariants. It will thus be seen that covariants and contravariants form two distinct and coextensive species of associated forms, which divide between them the wide and fertile empire of linear transformations so far as its provinces have been as yet laid open by the researches of analysts. In your next number I propose to enter much more largely into the subject generally. More particularly I shall describe the new method of Permutants, including the theory of Intermutants and Commutants (which latter are a species of the former, but embrace Determinants as a particular case), and their application to the theory of Invariants. I shall also exhibit the connexion between the theory of Invariants and that of Symmetrical Functions, and some remarkable theorems on Relative Invariants†.

Some of your readers may like to be informed that a Supplement to my last paper, under the title of "An Essay on Canonical Forms," has been since published‡; and that I have there given a much simpler method of solution of the problem of the reduction of quintic functions to their canonical form than in the original memoir, and extended the method successfully to the

of one from among the infinite variety of these forms to the discovery of the points of inflexion in a curve of the third order, in other words, to the Canonical Reduction of the Cubic Function of Three Letters, appears to have been the first occasion of their being turned to practical account.

* This is also true if I be taken any *covariant* instead of an invariant of the function.

† It will be readily apprehended that the definitions and conceptions above stated, respecting covariants and contravariants of two single functions, may be extended so as to comprehend systems of functions covariantive or contravariantive to one another.

‡ By Mr George Bell, University Bookseller, Fleet Street. [p. 203 below.]

reduction of all odd-degreed functions to their canonical form. I may take this occasion to state that the Lemma given in Note (B) of the Supplement, upon which this method of reduction is based, is an immediate deduction from the well-known theorem for the multiplication of Determinants.

There is a numerical error in "The Cubical Hyperdeterminant of the Twelfth Degree," worked out after the method of commutants by Mr Spottiswoode, given at the end of my paper in the May Number. The correct result will be stated in the next number of the *Journal*, where I hope also to be able to fix the number of distinct solutions of the problem of reducing a Sextic Function to its canonical form

$$u^6 + v^6 + w^6 + mu^2v^2w^2.$$

For odd-degreed functions there is never more than one solution possible, as shown in the Supplement referred to.

P.S. Since the above was sent to press, I have discovered an uniform mode of solution for the canonical reduction of functions, whether of odd or even degrees. The canonical form however, except for the fourth and eighth degrees, requires to be varied from that assumed in my previous paper. Thus, for the sixth degree the canonical form will be

$$au^6 + bv^6 + cw^6 + muvw\,(v - w)\,(w - u)\,(u - v),$$

where u, v, w are supposed to be connected by the identical equation $u + v + w = 0$. And there will be only *two* solutions—a remarkable and most unexpected discovery. For functions of the eighth degree there are five distinct solutions, and in general there is the strongest reason for believing (indeed it may be positively affirmed) that *when the canonical form has been rightly assumed* for a function of the even degree n, the number of solutions will be $\frac{1}{2}(n+2)$ when $\frac{1}{2}n$ is even, but $\frac{1}{4}(n+2)$ when $\frac{1}{2}n$ is odd. It turns out therefore that the theory for functions of the sixth degree is in some respects simpler than for those of the fourth. The investigation into canonical forms here referred to has led me to the discovery of a most unexpected theorem for finding all the invariants of a certain class, belonging to functions of two letters of an even degree.