34.

AN ESSAY ON CANONICAL FORMS, SUPPLEMENT TO A SKETCH OF A MEMOIR* ON ELIMINATION, TRANSFORMATION AND CANONICAL FORMS.

SINCE the above paper was in print I have succeeded in obtaining a canonical representation of the quadratic and cubic functions adjunctive to the general quintic (5th degreed) functions of two letters.

Let F the quintic function of x, y,

$$= u^5 + v^5 + w^5,$$

and

$$au + bv + cw = 0,$$

M being the modulus of the transformation, whereby transition is made from x, y to u, v. Then the quadratic adjunctive is

$$\frac{M^4}{c^4}\{a^4vw + b^4wu + c^4uv\};$$

and the cubic adjunctive is simply

$$\frac{1}{c^3}M^6(abc)^2 uvw\dagger.$$

Hence we can, in accordance with what I ventured to predict in the preceding sketch, find u, v, w, by means of a simple and practical co-process. To wit, call

$$F = lx^5 + 5mx^4y + 10nx^3y^2 + 10px^2y^3 + 5qxy^4 + ry^5.$$

[* p. 184 above. See p. 201, note ‡.]

† The knowledge of the existence of these *lower* adjunctive forms is mainly a consequence of Mr Cayley's splendid discovery of hyperdeterminant constants. In fact, they are respectively the quadratic and cubic hyperdeterminants in respect to ξ and η of $\frac{1}{1.2.3.4.5}\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^4 F$; x and y being treated as constants.

The fortunate proclaimer of a new outlying planet has been justly rewarded by the offer of a baronetcy and a national pension, which the writer of this wishes him long life and health to enjoy. In the meanwhile, what has been done in honour of the discoverer of a new and inexhaustible region of exquisite analysis?

Form the determinant

$$\begin{vmatrix} lx + my, & mx + ny, & nx + py \\ mx + ny, & nx + py, & px + qy \\ nx + py, & px + qy, & qx + ry \end{vmatrix}.$$

Let this cubic function, by solving it as a cubic equation, be made equal to

$$L(x + fy)(x + gy)(x + hy),$$

then

$$u = k(x + fy), \quad v = l(x + gy), \quad w = m(x + hy).$$

By means of the identity, $F = u^5 + v^5 + w^5$, l^5, m^5, n^5, are known by the solution of linear equations, and thus u, v, w, are determined by solving a cubic equation instead of one of the eighth degree, as in the method first given, and the process of canonising a quintic function is rendered *practically* possible.

For brevity sake let c represent unity. The constant determinant of the cubic adjunctive will be found to be

$$3M^{30}(abc)^{10}.$$

Calling, then, the cubic adjunctive of F, $C(F)$, we have the remarkable equation

$$uvw = \frac{C(F)}{\sqrt[5]{\{\frac{1}{3}\square C(F)\}}}.$$

It may also be shown that if we call the Hessian of F, $H(F)$, we shall have the following equally remarkable equation:

$$\square H(F) = \tfrac{1}{3}\square F \times \square C(F).$$

Again, calling the quadratic adjunctive of F, $Q(F)$, we shall easily find

$$\square Q(F) = M^{10}\left\{\begin{matrix} (a^{\frac{5}{2}} + b^{\frac{5}{2}} + c^{\frac{5}{2}}) \\ (a^{\frac{5}{2}} + b^{\frac{5}{2}} - c^{\frac{5}{2}}) \\ (a^{\frac{5}{2}} - b^{\frac{5}{2}} + c^{\frac{5}{2}}) \\ (a^{\frac{5}{2}} - b^{\frac{5}{2}} - c^{\frac{5}{2}}) \end{matrix}\right\},$$

or, if we please,

$$= M^{10}\left\{\begin{matrix} a^{10} + b^{10} + c^{10} \\ - 2a^5b^5 - 2a^5c^5 - 2b^5c^5 \end{matrix}\right\}.$$

When u, v, w are known, a, b, c, which are the resultants of v, w; w, u; u, v respectively are known. But their ratios, or, if we please to say so, the ratios of $a^5 : b^5 : c^5$, may be found independently and very elegantly as follows:—

Let $M^{10} \times$ product of the 4 forms of $a^{\frac{5}{2}} + 1^{\frac{1}{2}}b^{\frac{5}{2}} + 1^{\frac{1}{2}}c^{\frac{5}{2}} = A$,

$M^{20} \times$ product of the 16 forms of $a^{\frac{5}{4}} + 1^{\frac{1}{4}}b^{\frac{5}{4}} + 1^{\frac{1}{4}}c^{\frac{5}{4}} = B$,

$M^{30} \times a^{10} . b^{10} . c^{10} = C$.

A, B, C are known quantities, being respectively what we have called $\square Q(F)$, $\square(F)$*, $\frac{1}{3}\square C(F)$.

It may easily be shown that

$$B - A^2 = 128 M^{20} a^5 b^5 c^5 (a^5 + b^5 + c^5).$$

Hence $M^5 a^5$, $M^5 b^5$, $M^5 c^5$ are the roots of ρ in the cubic equation

$$\rho^3 + \frac{B - A^2}{2^7 C^{\frac{1}{2}}}\rho^2 + \frac{1}{4}\left\{\frac{(B - A^2)^2}{2^{14} C} - A\right\}\rho + C^{\frac{1}{2}} = 0.$$

A, B, C, it will be observed, are independent and, as they may be termed, prime or radical adjunctive constants. Hitherto much mystery and uncertainty have attached to the theory of hyperdeterminants, from its having been tacitly assumed that they were always either of lower dimensions than the ordinary determinant, or else algebraical functions of such, and of the determinant. Whereas we now see that, whilst the determinant of a function in two letters of the fifth degree is of eight dimensions, one of its radical or primitive hyperdeterminants is of four, but the other of twelve dimensions. This is a most valuable consequence, and would seem to indicate that the number of radical hyperdeterminants to a function, over and above the common determinant, is always equal to the number of parameters entering into its canonical form. The importance of this ascertainment of an unsuspected third *radical* constant, adjunctive to a quintic function of two letters, in making to march the theory of hyperdeterminants, can hardly be over-estimated.

From the equation last given we are enabled to assign the conditions in order that two functions of the fifth degree may be capable of being linearly transformed either into the other. For if we call F and F' two such linearly equivalent quintic functions, they must be capable each of being thrown under the same form $u^5 + v^5 + (lu + mv)^5$, where l and m shall be the same for each. Consequently we must have the roots of ρ in the same ratio for F and F', which conditions may be expressed by means of the two equations

$$\frac{B - A^2}{C^{\frac{2}{3}}} = \frac{B' - A'^2}{C'^{\frac{2}{3}}},$$

$$\frac{(B - A^2)^2 - 2^{14} AC}{C^{\frac{4}{3}}} = \frac{(B' - A'^2)^2 - 2^{14} A'C'}{C'^{\frac{4}{3}}},$$

* More strictly speaking (and this correction should be supplied throughout in the "Sketch"), B is the negative determinant of $\frac{1}{5}F$. After finding, by the method of characteristics, or any special artifices, the algebraic part of the value of a resultant or determinant, a process frequently of some complexity remains over in assigning its numerical multiplier; this part of the operation being analogous to that which occurs in the Integral Calculus, of determining the constant to be added after the general form of an integral has been determined. In the "Sketch," a correction for the numerical multiplier remains also to be applied to the expressions given for the successive Hessian determinants.

A', B', C', of course representing the same functions of the coefficients of F' as A, B, C, respectively of F.

The two conditions required in their simplest form are accordingly

$$\frac{A}{C^{\frac{1}{3}}} = \frac{A'}{C'^{\frac{1}{3}}},$$

$$\frac{B}{C^{\frac{2}{3}}} = \frac{B'}{C'^{\frac{2}{3}}},$$

or $$A^3 : B^2 : C :: A'^3 : B'^2 : C',$$

that is to say, *all quintic functions of two letters of which the determinant is to the subduplicate power of the radical hyperdeterminant of the twelfth order and to the sesquiduplicate power of the radical hyperdeterminant of the fourth order in given ratios, are mutually convertible.*

So for the quartic (that is, biquadratic) function of two letters, calling R and S the radical adjunctive constants of the second and third orders, the condition of convertibility between different forms of the same is, that $R^3 : S^2$ shall be a given ratio. And, in general, we may infer that the condition of convertibility between different functions of any degree is, that the several radical adjunctive constants of each raised respectively to such powers as will make them of like dimensions, shall be to one another in given ratios. Of course all cubic functions of two letters, according to this rule, are mutually convertible without any condition, they having but one radical adjunctive constant; and in fact all such functions, being representable as the sum of two cubes of new variables linearly related to those given, are necessarily convertible.

I have further succeeded in obtaining the canonical form of the *quadratic* adjunctive to *any odd degreed* function of two letters, which presents a wonderful analogy to the theory of relative determinants of *quadratic functions of any number* of letters, and constitutes an important step towards the construction of the theory of relative hyperdeterminants.

Let a function of two letters of the odd degree $m\,(=2n-1)$ be thrown under its canonical form,

$$u_1^m + u_2^m + \ldots + u_n^m,$$

and let there exist the $n-2$ equations,

$$a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = 0, \tag{1}$$

$$b_1 u_1 + b_2 u_2 + \ldots + b_n u_n = 0, \tag{2}$$

....

$$l_1 u_1 + l_2 u_2 + \ldots + l_n u_n = 0. \tag{n-2}$$

Then, if M be the modulus of the transformation which converts u_1, u_2 into

x, y, and if, on making θ_1, $\theta_2 \dots \theta_n$ disjunctively equal to $1, 2 \dots n$ we use (θ_{n-1}, θ_n) to denote in general the determinant

$$\begin{vmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_{n-2}} \\ b_{\theta_1}, & b_{\theta_2} \dots b_{\theta_{n-2}} \\ \dots & \dots \\ l_{\theta_1}, & l_{\theta_2} \dots l_{\theta_{n-2}} \end{vmatrix},$$

the quadratic adjunctive of $\dfrac{1}{m(m-1)\dots 2} F$ will be

$$\frac{M^{m-1}}{(1, 2)^{m-1}} \Sigma \{(\theta_r, \theta_s)^{m-1}(u_r \,.\, u_s)\}^*.$$

N.B. By means of this formula, and of the theorem for finding relative determinants of quadratic functions, we can obtain the general canonical form for one set of the biquadratic adjunctive constants (hyperdeterminants of the fourth order in Mr Cayley's language) of any odd degreed function of two letters †.

Thus, for the fifth degree, preserving the notation of the "Sketch," we have the biquadratic adjunctive constant

$$= \begin{vmatrix} 0, & c^4, & b^4, & a \\ c^4, & 0, & a^4, & b \\ b^4, & a^4, & 0, & c \\ a, & b, & c, & 0 \end{vmatrix} \times \frac{M^{10}}{c^{10}}.$$

For the seventh degree, if we suppose the function to be equal to

$$u^7 + v^7 + w^7 + \theta^7,$$

and

$$au + bv + cw + d\theta = 0,$$

$$a'u + b'v + c'w + d'\theta = 0;$$

the biquadratic adjunctive constant will be $\dfrac{M^{14}}{(cd' - c'd)^{14}}$ multiplied by the determinant

$$\begin{vmatrix} 0, & (ab' - a'b)^6, & (ac' - a'c)^6, & (ad' - a'd)^6, & a, & a' \\ (ba' - b'a)^6, & 0, & (bc' - b'c)^6, & (bd' - b'd)^6, & b, & b' \\ (ca' - c'a)^6, & (cb' - c'b)^6, & 0, & (cd' - c'd)^6, & c, & c' \\ (da' - d'a)^6, & (db' - d'b)^6, & (dc' - d'c)^6, & 0, & d, & d' \\ a, & b, & c, & d, & 0, & 0 \\ a', & b', & c', & d', & 0, & 0 \end{vmatrix}.$$

* The condition $m = 2n - 1$ is only necessary in order that $\Sigma_n'(u^m)$ may be a canonical, because a possible and determinate, form for any given function of the mth degree. But the theorem in the text, so far as it serves to obtain the quadratic adjunctive of $\Sigma_n'(u^m)$, is true for *all* odd values of m, whether greater or less than $2n - 1$.

† See Note (A) of Appendix.

The determinants of the Hessian, the post-Hessian, and the præter-post-Hessian of F will be found (in the case of the quintic function) to be always multiples of powers of the determinant of the given function, and of its cubic adjunctive; and I believe that in general for a function of two letters of any degree the determinants of all the derived forms in the Hessian scale*, will be necessarily algebraical functions of any two of them.

I hope very shortly to accomplish the reduction of functions, as high as the seventh degree of two letters, to their canonical form, and also to present a complete theory of the failing or singular cases of canonical forms.

Since the above was in print I have discovered the following

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for reducing a function of two letters of any odd degree to its canonical form.

Let the degree of the function be $(2n-1)$; then its canonical form is

$$u_1^{2n-1} + u_2^{2n-1} + \dots + u_n^{2n-1},$$

with $(n-2)$ linear relations between $u_1, u_2, \dots u_n$.

To find $u_1, u_2, \dots u_n$, proceed as follows. Let the given function of the $(2n-1)$th degree be supposed to be

$$a_1 x^{2n-1} + (2n-1)\, a_2 x^{2n-2} y + (2n-1)\frac{2n-2}{2}\, a_3 x^{2n-3} y^2 + \dots + a_{2n} y^{2n-1}.$$

Form the determinant

$$\begin{vmatrix} a_1x + a_2y, & a_2x + a_3y, & a_3x + a_4y \;\dots & a_nx + a_{n+1}y \\ a_2x + a_3y, & a_3x + a_4y, & \dots\dots\dots\dots & a_{n+1}x + a_{n+2}y \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_nx + a_{n+1}y, & a_{n+1}x + a_{n+2}y, & \dots\dots\dots & a_{2n-1}x + a_{2n}y \end{vmatrix}.$$

This determinant is a function of x and y of the nth degree, and by resolving an equation of the nth degree, may be decomposed into n factors, say

$$(l_1x + m_1y)(l_2x + m_2y) \dots (l_nx + m_ny);$$

* I use the term Hessian (more properly speaking the Boolian) Scale, to denote the *determinants* in respect of ξ and η of $\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \&c.\right)^2 F$.

Neither Hesse, however, nor any other writer up to the present time, had thought of constructing, and still less of turning to account, the functions (the first only excepted) which figure in this scale.

we shall then have

$$u_1 = p_1(l_1x + m_1y),$$
$$u_2 = p_2(l_2x + m_2y),$$
$$\dots\dots\dots\dots\dots\dots$$
$$u_n = p_n(l_nx + m_ny),$$

where the l's and m's are known, and the $(2n-1)$th powers of the p's may be found linearly, by means of the identical equation $\Sigma u^{2n-1} = F(x, y)$. Thus for example a function of the seventh degree of two letters may be reduced to its canonical form

$$(lx + my)^7 + (l'x + m'y)^7 + (l''x + m''y)^7 + (l'''x + m'''y)^7,$$

by the resolution of a biquadratic equation. My demonstration of this extraordinary and unexpected consequence rests upon the following lemma*, itself a very beautiful and striking theorem (no doubt capable of much generalisation) in the theory of determinants. Form the rectangular matrix consisting of n rows and $(n+1)$ columns

$$\begin{array}{llll} T_1, & T_2, & T_3 & \dots T_{n+1}, \\ T_2, & T_3, & T_4 & \dots T_{n+2}, \\ T_3, & T_4, & T_5 & \dots T_{n+3}, \\ \dots & \dots & \dots & \dots \\ T_n, & T_{n+1}, & T_{n+2} & \dots T_{2n}, \end{array}$$

where

$$T_i = a_1^{r-i}b_1^{s+i} + a_2^{r-i}b_2^{s+i} + \dots + a_{n-1}^{r-i}b_{n-1}^{s+i}.$$

Then all the $n+1$ determinants that can be formed by rejecting any *one* column at pleasure out of this matrix are identically zero.

In order the better to realise the proof, suppose

$$n = 4, \text{ so that } 2n - 1 = 7.$$

Let

$$F(x, y) = a_1x^7 + 7a_2x^6y + 21a_3x^5y^2 + 35a_4x^4y^3 + 35a_5x^3y^4$$
$$+ 21a_6x^2y^5 + 7a_7xy^6 + a_8y^7.$$

Suppose

$$t^7 + u^7 + v^7 + w^7 = F(x, y) = G(u, v),$$
$$at + bu = v,$$
$$a't + b'u = w.$$

Then, if M is the modulus of transition from x, y to u, v the hyper-

* See Note (B) of Appendix.

determinant, or, to adopt my new expression, the permutant P_4 (meaning thereby)

$$\begin{vmatrix} a_1x+a_2y, & a_2x+a_3y, & a_3x+a_4y, & a_4x+a_5y \\ a_2x+a_3y, & a_3x+a_4y, & a_4x+a_5y, & a_5x+a_6y \\ a_3x+a_4y, & a_4x+a_5y, & a_5x+a_6y, & a_6x+a_7y \\ a_4x+a_5y, & a_5x+a_6y, & a_6x+a_7y, & a_7x+a_8y \end{vmatrix},$$

which is a constant adjunctive in respect to ξ and η of $\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^6 F$, will, according to the principles laid down in the preceding "Sketch," be the product of a power of M multiplied by the corresponding adjunctive constant of $\left(\xi\frac{d}{du}+\eta\frac{d}{dv}\right)^6 G(u, v)$, and is therefore a multiple of the determinant

$$\begin{vmatrix} (1+A_1)t+A_2u, & A_2t+A_3u, & A_3t+A_4u, & A_4t+A_5u \\ A_2t+A_3u, & A_3t+A_4u, & A_4t+A_5u, & A_5t+A_6u \\ A_3t+A_4u, & A_4t+A_5u, & A_5t+A_6u, & A_6t+A_7u \\ A_4t+A_5u, & A_5t+A_6u, & A_6t+A_7u, & A_7t+(1+A_8)u \end{vmatrix},$$

where

$$A_1 = a^7 + a'^7, \quad A_2 = a^6b + a'^6b', \quad A_3 = a^5b^2 + a'^5b'^2 \ldots A_8 = b^7 + b'^7.$$

In this determinant the coefficient of u^4 is

$$\begin{vmatrix} A_2, & A_3, & A_4, & A_5 \\ A_3, & A_4, & A_5, & A_6 \\ A_4, & A_5, & A_6, & A_7 \\ A_5, & A_6, & A_7, & 1+A_8 \end{vmatrix}$$

which is numerically equal to

$$A_5\begin{vmatrix} A_3, & A_4, & A_5 \\ A_4, & A_5, & A_6 \\ A_5, & A_6, & A_7 \end{vmatrix} - A_6\begin{vmatrix} A_2, & A_4, & A_5 \\ A_3, & A_5, & A_6 \\ A_4, & A_6, & A_7 \end{vmatrix}$$

$$+A_7\begin{vmatrix} A_2, & A_3, & A_5 \\ A_3, & A_4, & A_6 \\ A_4, & A_5, & A_7 \end{vmatrix} - (1+A_8)\begin{vmatrix} A_2, & A_3, & A_4 \\ A_3, & A_4, & A_5 \\ A_4, & A_5, & A_6 \end{vmatrix}$$

$= 0$, because the second factors of the products are all zero by the lemma. Hence the permutant P_4 vanishes when $t = 0$, and consequently it contains t as a factor, and in like manner it may be proved to contain u, v, w.

Hence t, u, v, w are the algebraical factors of P_4, and precisely the same proof applies to show in the case of a function in x and y, say F_{2n-1}, of any odd degree $(2n-1)$ whatever, that the corresponding permutant P_n will contain the factors u_1, $u_2 \dots u_n$ linear functions of x, y, such that

$$u_1^{2n-1} + u_2^{2n-1} + \dots + u_n^{2n-1} = F_{2n-1}$$

as was to be shown.

Whenever P_n has equal roots, this will denote either (which is the more general case) that the usual canonical form fails and gives place to a singular form, (owing to some of the coefficients of transformation becoming infinite), or, which is the more special supposition, that the canonical form becomes catalectic by one or more of the linear roots* disappearing. Thus in the cubic function, if P_2 has equal roots, and consequently its determinant (which is coincident with that of the function itself) vanish, then the canonical form in general fails; so that, for example, $ax^3 + bx^2y$ cannot in general be exhibited as the sum of two cubes: if, however, certain further relations obtain between the coefficients of F, the canonical form reappears catalectically, the function becoming in fact representable as a single cube. So, again, for the quintic function (referring back to the notation above [page 205]), if P_3 have equal roots, that is if $C=0$, the canonical form fails, unless at the same time $B - A^2 = 0$, in which case the function becomes the sum of two fifth powers; but if furthermore $A=0$, then this catalectic form again gives place to a singular form, which, on the satisfaction of a further condition between the coefficients, again in its turn gives way before a (bicatalectic, that is) doubly catalectic form, namely, a single fifth power.

It is remarkable, that the form to which Mr Jerrard's method reduces the function of the fifth degree, expressed homogeneously as $ax^5 + bxy^4 + cy^5$, is a singular form, being incapable of being exhibited as the sum of three cubes; such, however, is not the case with the form $ax^5 + bx^3y^2 + cy^5$. It may further be remarked, that although the singly catalectic form of the quintic function is expressible by two conditions only, namely, $C=0$, $B-A^2=0$, it will be indicated by P_3 (which being a cubic function of x and y contains four terms) completely disappearing, so that apparently four conditions would appear to be required or implied. But of course these must be capable of being shown to be non-independent, and to be merely tantamount to the two independent ones, $C=0$, $B-A^2=0$. The theory of the catalectic forms of functions of the higher degrees of two variables presents many strong points of resemblance and of contrast to that of the catalectic forms of quadratic functions of several variables.

One important and immediate corollary from the General Theorem is, that the constants which enter into the linear functions appurtenant to the canonical form of any function of an odd degree form a single and unique system; or, in other words, the canonical forms for such functions are void of

* u_1, $u_2 \dots u_n$ may be termed the linear roots of the form F_{2n-1}.

multiplicity, a result contrary to what might have been anticipated, and to what we know is the case for the canonical forms of functions of an even degree.

It may further be shown that if we have the $(n-2)$ equations

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0,$$
$$b_1u_1 + b_2u_2 + \ldots + b_nu_n = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$l_1u_1 + l_2u_2 + \ldots + l_nu_n = 0,$$

and call M the modulus of transformation in respect to u_1, u_2, and if we make

$$P_n = Ku_1u_2 \ldots u_n,$$

then

$$\left|\begin{matrix} a_3, & a_4 \ldots a_n \\ b_3, & b_4 \ldots b_n \\ \ldots & \ldots \\ l_3, & l_4 \ldots l_n \end{matrix}\right|^{\frac{1}{2}n(n-1)} K$$

is equal to the product of the $\frac{1}{2}n(n-1)$ factors of the form

$$\left|\begin{matrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_{n-2}} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_{n-2}} \\ \ldots & \ldots \\ l_{\theta_1}, & l_{\theta_2} \ldots l_{\theta_{n-2}} \end{matrix}\right|,$$

θ_1, $\theta_2 \ldots \theta_{n-2}$ being any $(n-2)$ numbers out of the n numbers 1, 2, 3 … n.

It may hence be shown that

$$u_1u_2 \ldots u_n = \frac{P_n}{\left(\frac{1}{m} \underset{x,y}{\square} P_n\right)^{\frac{1}{2n-1}}}\text{*},$$

m being a number which is a function of n, and which may be shown to be equal to $\underset{x,y}{\square}(x^{n-1}y + xy^{n-1}) \div$ product of the squared differences of the roots of $l^{n-2} = 1$, that is

$$m = \left\{\frac{(n-1)^2 - 1}{-(n-2)}\right\}^{n-2} = (-n)^{n-2},$$

and thus

$$u_1u_2 \ldots u_n = \frac{P_n}{\sqrt[2n-1]{\{(-n)^{-(n-2)} \cdot \underset{x,y}{\square} P_n\}}}.$$

* $\underset{x,y}{\square}$ means the determinant in respect to x and y.

As an example of the mode of finding $u_1, u_2 \dots u_n$, let

$$F = 3x^5 + 20x^3y^2 + 10xy^4,$$

then

$$P_3 = \begin{vmatrix} 3x, & 2y, & 2x \\ 2y, & 2x, & 2y \\ 2x, & 2y, & 2x \end{vmatrix} = 4x^3 - 4y^2x.$$

Hence

$$u = fx, \quad v = g(x+y), \quad w = h(x-y).$$

To find f, g, h, we have $u^5 + v^5 + w^5 = F$, hence

$$f^5 + g^5 + h^5 = 3; \quad g^5 + h^5 = 2; \quad g^5 - h^5 = 1;$$

whence we have

$$F = x^5 + (x+y)^5 + (x-y)^5.$$

Again, we find

$$\square\,(4x^3 - 4y^2x) = -4^4 \times 12,$$

$$\left(\frac{1}{(-3)}\,\square\,P_3\right)^{\frac{1}{3}} = 4,$$

and accordingly

$$x(x+y)(x-y) = \frac{P_3}{\{(-3)^{-1}.\,\square\,P_3\}^{\frac{1}{3}}},$$

according to the general formula above given.

As a second example let

$$F = 3x^7 + 42x^5y^2 + 70x^3y^4 + 14xy^6 + y^7;$$

then

$$P_4 = \begin{vmatrix} 3x, & 2y, & 2x, & 2y \\ 2y, & 2x, & 2y, & 2x \\ 2x, & 2y, & 2x, & 2y \\ 2y, & 2x, & 2y, & 2x+y \end{vmatrix} = 4(x^3y - xy^3) = 4xy(x-y)(x+y),$$

and accordingly we shall find

$$x^7 + y^7 + (x-y)^7 + (x+y)^7 = F.$$

Moreover

$$\square\,(4x^3y - 4xy^3) = 4^9,$$

and

$$4^{4-2} \qquad = 4^2.$$

Thus

$$\frac{P_4}{tuvw} = \sqrt[7]{\frac{\square\,P_4}{4^{4-2}}},$$

agreeable to the general formula.

As a corollary to our general proposition, it may be remarked, that if F_{2n-1} be a symmetrical function of x, y of the $(2n-1)$th degree, $P_n(F_{2n-1})$ will be also a symmetrical function of x and y, and may therefore be resolved into its factors by solving a *recurring* equation of the nth degree, which may, by well-known methods, be made to depend on the solution of an equation of the $\frac{1}{2}n$th or $\frac{1}{2}(n-1)$th degree, according as n is even or odd.

Hence the reduction of a function of two letters of the degree $4m \pm 1$ to its canonical form as the sum of powers may be made to depend on the solution of an equation of the mth degree; so that, for example, a symmetrical function of x, y, as high as the fifteenth or seventeenth degree, may be reduced by means of a biquadratic equation only.

In a short time I hope to present to the public a complete solution of the canonical forms of functions of two letters of even degrees, and possibly to exhibit some important applications of the principles of the method to the theory of numbers.

APPENDIX.

NOTE (A).

The permutants (meaning, in Mr Cayley's language, the hyperdeterminants) of $F_{2n+1}(x, y)$ of the fourth dimension in respect to the coefficients of F, may be all obtained by taking the quadratic permutant in respect to x and y of the quadratic permutant in respect of ξ and η of

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{2l} F_{2n+1}(x, y),$$

l having any integer value from 1 to n.

In extension of a theorem in the foregoing Supplement, which applies only to the case of $l=n$, I am able to state the following more general theorem, in which the same notation is preserved as above [page 207]. The quadratic permutant in respect to ξ and η of

$$\frac{1}{(2n+1)\,2n\ldots(2n-2l+2)}\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{2l} F_{2n+1}(x, y),$$

is equal to

$$\frac{M^{2l}}{(1,\,2)^{2l}}\,\Sigma\left\{(\theta_r,\ \theta_s)^{2l}(u_r\,.\,u_s)^{2n+1-2l}\right\}.$$

If now we proceed to form the quadratic permutant of the above sum in respect to x and y, we know *à priori*, by reason of Mr Cayley's invaluable researches, that we shall not get radically distinct results for all values, but only for certain periodically changing values of l.

I have not yet had leisure to seek for an explicit demonstration of this remarkable law, founded upon the above given canonical representation.

Note (B).

The lemma, upon which the general method for reducing odd degreed functions to their canonical form is founded, may be stated rather more simply and more generally as follows :—

The determinant

$$\begin{vmatrix} T_{r_1}, & T_{r_2} & \ldots T_{r_n} \\ T_{r_1+l_1}, & T_{r_2+l_1} & \ldots T_{r_n+l_1} \\ T_{r_1+l_2}, & T_{r_2+l_2} & \ldots T_{r_n+l_2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ T_{r_1+l_{n-1}}, & T_{r_2+l_{n-1}} & \ldots T_{r_n+l_{n-1}} \end{vmatrix},$$

where T_θ denotes $A_1a_1{}^\theta + A_2a_2{}^\theta + \ldots + A_ma_m{}^\theta$ *provided that m is less than n*, is identically zero. In the theorem, as thus stated, there is no substantial loss of generality arising from the omission of the b's.

Thus stated the theorem and its extensions evidently repose upon the same or the like basis as the theory of partial fractions.

Note (C), *referring to the original "Sketch."*

The Boolo-Hessian scale of determinants furnishes a very pretty general theorem of geometrical reciprocity in connexion with the doctrine of successive polars. Let $F(x, y, z)$, a cubic homogeneous function of x, y, z equated to zero, express in general a curve of the third degree; then $\left(a\frac{d}{dx}+b\frac{d}{dy}+c\frac{d}{dz}\right)F$ will express its first polar in respect to the point a, b, c, that is, the conic which passes through the six points in which the tangents drawn from a, b, c to touch the given curve meet the same.

Again, if we take l, m, n the coordinates of any new point,

$$\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\left(a\frac{d}{dx}+b\frac{d}{dy}+c\frac{d}{dz}\right)F$$

will express the polar, that is the chord of contact of the above conic, in respect to the last named point. If now we eliminate l, m, n between the three equations

$$\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\left(a\frac{d}{dx}+b\frac{d}{dy}+c\frac{d}{dz}\right)F=0,$$

$$\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\left(a'\frac{d}{dx}+b'\frac{d}{dy}+c'\frac{d}{dz}\right)F=0,$$

$$\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\left(a''\frac{d}{dx}+b''\frac{d}{dy}+c''\frac{d}{dz}\right)F=0,$$

it is easily seen that the resultant of the elimination is the square of the determinant

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

multiplied by the Hessian of the given function. And, moreover, that if we eliminate x, y, z we shall obtain precisely the same result with the letters l, m, n substituted for x, y, z. Hence it follows, that if we take the doubly infinite system of first polars to a given curve of the third degree, in respect to all the points lying in its plane, and then from any point in the *Hessian* to the given curve, draw pairs of tangents to each conic of the system so generated, then all the chords of contact will meet in one and the same point, which will itself be also a point situated upon the Hessian and conjugate to the former.

So, in general, for a function of any degree of any number of letters, viewed with relation to the doctrine of successive polars, the determinants of the Boolo-Hessian scale take one another up in pairs; namely the first takes up the last but one, the second the last but two, and so on; and consequently, if the degree of the function be odd, that function which (making abstraction of the constant determinant at the end) lies in the middle of the scale pairs with itself, and, in a sense analogous to that above exhibited for a function of the third degree, may be said to be always its own reciprocal.

P.S. I have just discovered the method of reducing functions of two letters of *even degrees* to their canonical form, which will shortly be published in a second Supplement.

At present I offer the annexed theorem (which strikingly contrasts with the law of uniqueness demonstrated of functions of an odd degree) as a foretaste of the enchanting developments with which I hope shortly to present my readers:—

If a given homogeneous function of x and y of the degree $2n$ be supposed to be thrown under its canonical form,

$$u_1^{2n} + u_2^{2n} + \ldots + u_n^{2n} + K(u_1 u_2 \ldots u_n)^2,$$

then will K^n have $n^2 - 1$ in general distinct values, to each of which will correspond a single distinct system of the linear functions of x and y,

$$1^{\frac{1}{n}} u_1, \quad 1^{\frac{1}{n}} u_2, \ldots \; 1^{\frac{1}{n}} u_n.$$