1.

ANALYTICAL DEVELOPMENT OF FRESNEL'S OPTICAL THEORY OF CRYSTALS.

[*Philosophical Magazine*, XI. (1837), pp. 461—469, 537—541; XII. (1838), pp. 73—83, 341—345.]

THE following is, I believe, the first successful attempt to obtain the full development of Fresnel's Theory of Crystals by direct geometrical methods. Hitherto little has been done beyond finding and investigating the properties of the wave surface, a subject certainly curious and interesting, but not of chief importance for ordinary practical purposes. Mr Kelland, in a most valuable contribution to the *Cambridge Philosophical Transactions**, has incidentally obtained the difference of the squares of the velocities of a plane front in terms of the angles made by it with the optic axes. I have obtained each of the velocities *separately*, and in a form precisely the same for biaxal as for uniaxal crystals.

I have also assigned in my last proposition the place of the lines of vibration in terms of the like quantities, and *that* in a shape remarkably convenient for determining the *plane* of polarization when the ray is given. For at first sight there appears to be some ambiguity in selecting *which* of the *two* lines of vibration is to be chosen when the front is known. If p be the perpendicular from the centre of the surface of elasticity let fall upon the front, ι_1, ι_2 the angles made by the front with the optic planes, ϵ_1, ϵ_2 the angles between its *due* line of vibration and the optic axes, I have shown that

$$\cos \epsilon_1 = \sqrt{\left(\frac{b^2 - p^2}{a^2 - c^2} \cdot \frac{\sin \iota_1}{\sin \iota_2}\right)}, \quad \cos \epsilon_2 = \sqrt{\left(\frac{b^2 - p^2}{a^2 - c^2} \cdot \frac{\sin \iota_2}{\sin \iota_1}\right)},$$

so that all doubt is completely removed. The equation preparatory to obtaining the wave surface is found in Prop. 6 by common algebra, without any use of the properties of maxima and minima, and various other curious relations are discussed.

Without the most careful attention to preserve pure symmetry, the expressions could never have been reduced to their present simple forms.

* See *Lond. and Edinb. Phil. Mag.* Vol. x. p. 336.

Analytical Reduction of Fresnel's Optical Theory of Crystals.

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In Proposition 1, a plane front within a crystal being given, the two lines of vibration are investigated.

In Proposition 2 it is shown that the product of the cosines of the inclinations of one of the axes of elasticity to the two lines of vibration, is to the same for either other axis of elasticity in a constant ratio for the same crystal; and the two lines of vibration are proved to be perpendicular to each other.

In Proposition 3, a line of vibration being given, the front to which it belongs is determined; and it is proved that there is only one such, and consequently any line of vibration has but one other line conjugate to it.

In Proposition 4, certain relations are instituted between the positions of, and velocities due to, conjugate lines.

In Proposition 5, the angles made by the front with the planes of elasticity are found in terms of the velocities only.

In Proposition 6, the above is reversed.

In Proposition 7, the position of the planes in which the two velocities are equal (viz. the optic planes) is determined.

In Proposition 8, the position of a front in respect to the optic axes is expressed in terms of the velocities.

In Proposition 9, the problem is reversed, and it is shown that if v_1, v_2 be the two normal velocities with which any front can move perpendicular to itself, and ι_1, ι_2 the angles which it makes with the optic planes, then

$$v_1^2 = a^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2,$$

$$v_2^2 = a^2\left(\sin\frac{\iota_1-\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1-\iota_2}{2}\right)^2.$$

In the 10th the angle made by a line of vibration with the axes of elasticity is expressed in terms of the two velocities of the front to which it belongs.

In the 11th Proposition the velocity due to any line of vibration is expressed in terms of the angles which it makes with the optic axes, viz.

$$v^2 - b^2 = (a^2 - c^2)\cos\epsilon_1\cos\epsilon_2.$$

In the 12th Proposition ϵ_1, ϵ_2 are separately expressed in terms of ι_1, ι_2.

In the Appendix I have given the polar or rather radio-angular equation to the wave surface, from which the celebrated proposition of the ray flows as an immediate consequence.

PROPOSITION 1.

If $$lx + my + nz = 0 \tag{a}$$

be the equation to a given front, to determine the lines of vibration therein.

It is clear that if x, y, z be any point in one of these lines, the force acting on a particle placed there when resolved into the plane must tend to the centre. Consequently the line of force at x, y, z must meet the perpendicular drawn upon the front from the origin. Now the equation to this perpendicular is

$$\frac{X}{l} = \frac{Y}{m} = \frac{Z}{n} \tag{1}$$

and the forces acting at x, y, z are a^2x, b^2y, c^2z parallel to x, y, z, so that the equation to the line of force is

$$\frac{X - x}{a^2x} = \frac{Y - y}{b^2y} = \frac{Z - z}{c^2z}. \tag{2}$$

From (2) we obtain

$$b^2yX - a^2xY = (b^2 - a^2)\,xy \tag{3}$$

$$c^2zY - b^2yZ = (c^2 - b^2)\,yz \tag{4}$$

$$a^2xZ - c^2zX = (a^2 - c^2)\,zx. \tag{5}$$

Hence

$$(b^2 - a^2)\,xyn + (c^2 - b^2)\,yzl + (a^2 - c^2)\,zxm$$
$$= b^2y\,(nX - lZ) + c^2z\,(lY - mX) + a^2x\,(mZ - nY);$$

but by equations (1)

$$lZ - nX = 0,\quad mX - lY = 0,\quad nY - mZ = 0$$

therefore

$$(b^2 - a^2)\frac{n}{z} + (c^2 - b^2)\frac{l}{x} + (a^2 - c^2)\frac{m}{y} = 0. \tag{b}$$

Also we have

$$nz + lx + my = 0 \tag{a}$$

therefore

$$(b^2 - a^2)\,n^2 + (c^2 - b^2)\,l^2 + nl\left((c^2 - b^2)\frac{z}{x} + (b^2 - a^2)\frac{x}{z}\right) = (a^2 - c^2)\,m^2$$

or

$$(c^2 - b^2)\left(\frac{z}{x}\right)^2 + \frac{1}{nl}\left\{(c^2 - b^2)\,l^2 + (b^2 - a^2)\,n^2 - (a^2 - c^2)\,m^2\right\}\frac{z}{x} + (b^2 - a^2) = 0.$$

And in like manner interchanging b, y, m with c, z, n

$$(b^2 - c^2)\left(\frac{y}{x}\right)^2 + \frac{1}{ml}\left\{(b^2 - c^2)\,l^2 + (c^2 - a^2)\,m^2 - (a^2 - b^2)\,n^2\right\}\frac{y}{x} + (c^2 - a^2) = 0.$$

Hence if $\left(\frac{y_1}{x_1}, \frac{z_1}{x_1}\right)\left(\frac{y_2}{x_2}, \frac{z_2}{x_2}\right)$ be the two systems of values of $\frac{y}{x}, \frac{z}{x}$, then

$$\left(\frac{Y}{X}=\frac{y_1}{x_1},\ \frac{Z}{X}=\frac{z_1}{x_1}\right)\left(\frac{Y}{X}=\frac{y_2}{x_2},\ \frac{Z}{X}=\frac{z_2}{x_2}\right)$$

are the two lines of vibration required.

Proposition 2.

By last proposition it appears that

$$\frac{y_1y_2}{x_1x_2}=\frac{c^2-a^2}{b^2-c^2} \qquad (c)$$

and

$$\frac{z_1z_2}{x_1x_2}=\frac{b^2-a^2}{c^2-b^2} \qquad (d)$$

therefore

$$\frac{y_1y_2+z_1z_2}{x_1x_2}=\frac{c^2-b^2}{b^2-c^2}=-1$$

therefore

$$x_1x_2+y_1y_2+z_1z_2=0.$$

And therefore the two lines of vibration are perpendicular to each other.

N.B. Equations (c) and (d) must not be overlooked.

Proposition 3.

A line of vibration is given $\left(\textit{that is } \frac{y_1}{x_1}, \frac{z_1}{x_1} \textit{ are given}\right)$ *and the position of the front is to be determined.*

Let $lx+my+nz=0$ be the front required, then $lx_1+my_1+nz_1=0$, and

$$(b^2-c^2)\frac{l}{x_1}+(c^2-a^2)\frac{m}{y_1}+(a^2-b^2)\frac{n}{z_1}=0.$$

Eliminating n we get

$$l\left((a^2-b^2)\frac{x_1}{z_1}-(b^2-c^2)\frac{z_1}{x_1}\right)+m\left((a^2-b^2)\frac{y_1}{z_1}-(c^2-a^2)\frac{z_1}{y_1}\right)=0$$

therefore

$$\frac{l}{m}=\frac{x_1}{y_1}\frac{(a^2-b^2)\,y_1^2-(c^2-a^2)\,z_1^2}{(b^2-c^2)\,z_1^2-(a^2-b^2)\,x_1^2}$$

$$=\frac{x_1}{y_1}\frac{a^2(x_1^2+y_1^2+z_1^2)-(a^2x_1^2+b^2y_1^2+c^2z_1^2)}{b^2(x_1^2+y_1^2+z_1^2)-(a^2x_1^2+b^2y_1^2+c^2z_1^2)}.$$

If now we make $x_1^2 + y_1^2 + z_1^2 = 1$

$$a^2x_1^2 + b^2y_1^2 + c^2z_1^2 = v_1^2$$

and therefore

$$\frac{l}{m} = \frac{x_1}{y_1} \cdot \frac{a^2 - v_1^2}{b^2 - v_1^2}$$

and in like manner

$$\frac{l}{n} = \frac{x_1}{z_1} \cdot \frac{a^2 - v_1^2}{c^2 - v_1^2};$$

therefore

$$(a^2 - v_1^2)\, x_1x + (b^2 - v_1^2)\, y_1y + (c^2 - v_1^2)\, z_1z = 0$$

is the equation required.

Proposition 4.

$\frac{l}{m}, \frac{l}{n}$ having each only one value, shows that only one front corresponds to the given line of vibration. Let x_2, y_2, z_2, v_2 correspond to x_1, y_1, z_1, v_1 for the conjugate line of vibration, then the equation to the front may be expressed likewise by

$$(a^2 - v_2^2)\, x_2x + (b^2 - v_2^2)\, y_2y + (c^2 - v_2^2)\, z_2z = 0,$$

so that

$$\frac{(a^2 - v_1^2)\, x_1}{(a^2 - v_2^2)\, x_2} = \frac{(b^2 - v_1^2)\, y_1}{(b^2 - v_2^2)\, y_2} = \frac{(c^2 - v_1^2)\, z_1}{(c^2 - v_2^2)\, z_2}.$$

Proposition 5.

To find ω, ϕ, ψ, the angles made by the front with the planes of elasticity in terms of v_1, v_2.

By the last proposition

$$(\cos\omega)^2 = \frac{(a^2 - v_1^2)^2\, x_1^2}{(a^2 - v_1^2)^2\, x_1^2 + (b^2 - v_1^2)^2\, y_1^2 + (c^2 - v_1^2)^2\, z_1^2}$$

$$= \frac{(a^2 - v_1^2)(a^2 - v_2^2)\, x_1x_2}{(a^2 - v_1^2)(a^2 - v_2^2)\, x_1x_2 + (b^2 - v_1^2)(b^2 - v_2^2)\, y_1y_2 + (c^2 - v_1^2)(c^2 - v_2^2)\, z_1z_2}.$$

Now, by Proposition 2,

$$\frac{x_1x_2}{c^2 - b^2} = \frac{y_1y_2}{a^2 - c^2} = \frac{z_1z_2}{b^2 - a^2}$$

therefore $(\cos\omega)^2$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)}{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)+(b^2-v_1^2)(b^2-v_2^2)(a^2-c^2)+(c^2-v_1^2)(c^2-v_2^2)(b^2-a^2)}$$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)}{a^4(c^2-b^2)+b^4(a^2-c^2)+c^4(a^2-b^2)}$$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}.$$

Similarly,

$$(\cos\phi)^2=\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)},$$

$$(\cos\psi)^2=\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}.$$

Proposition 6.

To find v_1, v_2 in terms of ω, ϕ, ψ.

By the last proposition

$$\frac{(\cos\omega)^2}{a^2-v_1^2}=\frac{a^2}{(a^2-b^2)(a^2-c^2)}-v_2^2\cdot\frac{1}{(a^2-b^2)(a^2-c^2)}$$

$$\frac{(\cos\phi)^2}{b^2-v_1^2}=\frac{b^2}{(b^2-a^2)(b^2-c^2)}-v_2^2\cdot\frac{1}{(b^2-a^2)(b^2-c^2)}$$

$$\frac{(\cos\psi)^2}{c^2-v_1^2}=\frac{c^2}{(c^2-b^2)(c^2-a^2)}-v_2^2\cdot\frac{1}{(c^2-a^2)(c^2-b^2)}$$

therefore

$$\frac{(\cos\omega)^2}{a^2-v_1^2}+\frac{(\cos\phi)^2}{b^2-v_1^2}+\frac{(\cos\psi)^2}{c^2-v_1^2}=0.$$

Just in the same way

$$\frac{(\cos\omega)^2}{a^2-v_2^2}+\frac{(\cos\phi)^2}{b^2-v_2^2}+\frac{(\cos\psi)^2}{c^2-v_2^2}=0,$$

so that v_1^2, v_2^2 are the two roots of the equation

$$\frac{(\cos\omega)^2}{a^2-v^2}+\frac{(\cos\phi)^2}{b^2-v^2}+\frac{(\cos\psi)^2}{c^2-v^2}=0.$$

Cor. Hence the equation to the wave surface may be obtained by making

$$(\cos\omega)x+(\cos\phi)y+(\cos\psi)z=v,$$

or if we please to apply Prop. 5, we may make

$$\sqrt{\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}}\,.\,x+\sqrt{\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)}}\,.\,y$$

$$+\sqrt{\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}}\,.\,z=v_1,$$

or, if we please*,

$$\sqrt{\frac{(a^2u^2-1)(a^2-v^2)}{(a^2-b^2)(a^2-c^2)}}\,.\,x+\sqrt{\frac{(b^2u^2-1)(b^2-v^2)}{(b^2-a^2)(b^2-c^2)}}\,.\,y$$

$$+\sqrt{\frac{(c^2u^2-1)(c^2-v^2)}{(c^2-a^2)(c^2-b^2)}}\,.\,z=1.$$

Proposition 7.

To find when $v_1=v_2$.

By Prop. 4,

$$\frac{x_1(v_1^2-a^2)}{x_2(v_2^2-a^2)}=\frac{y_1(v_1^2-b^2)}{y_2(v_2^2-b^2)}=\frac{z_1(v_1^2-c^2)}{z_2(v_2^2-c^2)}. \qquad (\theta)$$

Hence when $v_1=v_2$ we have, generally speaking,

$$\frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}.$$

Now $$x_1x_2+y_1y_2+z_1z_2=0;$$

therefore $x_1^2+y_1^2+z_1^2$ would $=0$, which is absurd.

The only case therefore when v_1 can $=v_2$ is when one of those terms of equation (θ) becomes $\frac{0}{0}$: thus suppose $v_1=b$, then we have $\frac{x_1}{x_2}=\frac{z_1}{z_2}=\frac{0}{0}$, and we can no longer infer $\frac{x_1}{x_2}=\frac{y_1}{y_2}$.

Let now $(\omega_1,\phi_1,\psi_1)(\omega_2,\phi_2,\psi_2)$ be the two systems of values which ω, ϕ, ψ assume when $v_1=v_2=b$, then applying the equation of Prop. 5 we have

$$\cos\omega_1=\sqrt{\frac{a^2-b^2}{a^2-c^2}} \qquad \cos\omega_2=\sqrt{\frac{a^2-b^2}{a^2-c^2}}$$

$$\cos\phi_1=0 \qquad \cos\phi_2=0$$

$$\cos\psi_1=\sqrt{\frac{b^2-c^2}{a^2-c^2}} \qquad \cos\psi_2=\sqrt{\frac{b^2-c^2}{a^2-c^2}},$$

so that b must correspond to the mean axis.

[* See below, p. 27. Ed.]

PROPOSITION 8.

ι_1, ι_2 being the angles made by the front with the optic planes, to find ι_1, ι_2 in terms of v_1, v_2.

By analytical geometry

$$\cos\iota_1 = \cos\omega \,.\, \cos\omega_1 + \cos\phi \,.\, \cos\phi_1 + \cos\psi \,.\, \cos\psi_1$$

$$= \sqrt{\frac{(v_1^2 - a^2)(v_2^2 - a^2)}{(a^2 - b^2)(a^2 - c^2)}} \,.\, \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$$

$$+ \sqrt{\frac{(v_1^2 - c^2)(v_2^2 - c^2)}{(c^2 - a^2)(c^2 - b^2)}} \,.\, \sqrt{\frac{c^2 - b^2}{c^2 - a^2}}$$

$$= \frac{\sqrt{\{(v_1^2 - a^2)(v_2^2 - a^2)\}} + \sqrt{\{(v_1^2 - c^2)(v_2^2 - c^2)\}}}{a^2 - c^2},$$

and similarly

$$\cos\iota_2 = \cos\omega \,.\, \cos\omega_2 + \cos\phi \,.\, \cos\phi_2 + \cos\psi \,.\, \cos\psi_2$$

$$= \frac{\sqrt{\{(v_1^2 - a^2)(v_2^2 - a^2)\}} - \sqrt{\{(v_1^2 - c^2)(v_2^2 - c^2)\}}}{a^2 - c^2}.$$

PROPOSITION 9.

To find v_1, v_2 in terms of ι_1, ι_2.

By the last proposition

$$\cos\iota_1 \,.\, \cos\iota_2 = \frac{(v_1^2 - a^2)(v_2^2 - a^2) - (v_1^2 - c^2)(v_2^2 - c^2)}{(a^2 - c^2)^2}$$

$$= \frac{(a^4 - c^4) - (a^2 - c^2)(v_1^2 + v_2^2)}{(a^2 - c^2)^2}$$

$$= \frac{(a^2 + c^2) - (v_1^2 + v_2^2)}{(a^2 - c^2)}$$

therefore

$$v_1^2 + v_2^2 = a^2 + c^2 - (a^2 - c^2)\cos\iota_1 \cos\iota_2.$$

Again, $(\sin\iota_1)^2 \,.\, (\sin\iota_2)^2 = 1 - (\cos\iota_1)^2 - (\cos\iota_2)^2 + (\cos\iota_1)^2 (\cos\iota_2)^2$

$$= 1 - 2 \,.\, \frac{(v_1^2 - a^2)(v_2^2 - a^2) + (v_1^2 - c^2)(v_2^2 - c^2)}{(a^2 - c^2)^2}$$

$$+ \frac{(a^2 + c^2)^2 - 2(a^2 + c^2)(v_1^2 + v_2^2) + (v_1^2 + v_2^2)^2}{(a^2 - c^2)^2}$$

$$= \frac{v_1^4 - 2v_1^2 v_2^2 + v_2^4}{(a^2 - c^2)^2}$$

therefore

$$v_1^2 - v_2^2 = (a^2 - c^2) \sin \iota_1 . \sin \iota_2$$

but
$$v_1^2 + v_2^2 = (a^2 + c^2) - (a^2 - c^2) \cos \iota_1 \cos \iota_2$$

therefore

$$v_1^2 = \frac{a^2 + c^2}{2} - \frac{a^2 - c^2}{2} \cos (\iota_1 + \iota_2)$$

$$= a^2 \left(\sin \frac{\iota_1 + \iota_2}{2}\right)^2 + c^2 \left(\cos \frac{\iota_1 + \iota_2}{2}\right)^2$$

$$v_2^2 = \frac{a^2 + c^2}{2} - \frac{a^2 - c^2}{2} \cos (\iota_1 - \iota_2)$$

$$= a^2 \left(\sin \frac{\iota_1 - \iota_2}{2}\right)^2 + c^2 \left(\cos \frac{\iota_1 - \iota_2}{2}\right)^2.$$

Thus for uniaxal crystals where $\iota_1 + \iota_2 = 180°$

$$v_1^2 = a^2$$

$$v_2^2 = a^2 (\cos \iota)^2 + c^2 (\sin \iota)^2.$$

COR. Hence we may reduce the discovery of the two fronts into which a plane front is refracted on entering a crystal to the following trigonometrical problem.

Let a sphere be described about any point in the line in which the air front intersects the plane of incidence. Let the great circle PI denote the latter plane, IF the former, OA, OC also great circles, the planes of single velocity. Suppose IGH to be one of the refracted fronts intersecting OA, OC in G and H, then

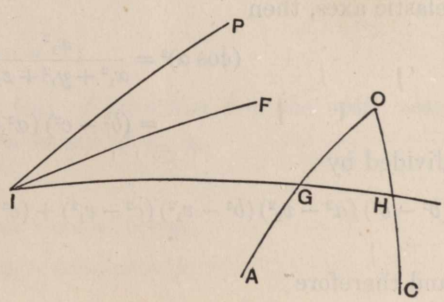

Fig. 1.

$$\frac{(a^2 + c^2) - (a^2 - c^2) \cos (G + H)}{2 (\text{vel. in air})^2} = \frac{(\sin PIF)^2}{(\sin PIGH)^2}.$$

The double sign will give rise to two positions of the refracted front IGH.

The propositions which follow are perhaps more curious than immediately useful.

Proposition 10.

To determine the portion of a line of vibration in terms of the two velocities of its corresponding front.

We have here to determine the quantities $\frac{y_1}{x_1}$, $\frac{z_1}{x_1}$ (of Prop. 1) in terms of v_1, v_2, or on putting $x_1^2+y_1^2+z_1^2=1$, x_1, y_1, z_1 are to be found in terms of v_1, v_2.

By Prop. 3

$$x_1 : y_1 : z_1 :: \frac{l}{a^2-v_1^2} : \frac{m}{b^2-v_1^2} : \frac{n}{c^2-v_1^2}$$

and by Prop. 5

$$\begin{aligned} l^2 : m^2 : n^2 :: &(b^2-c^2)(a^2-v_1^2)(a^2-v_2^2) \\ :&(c^2-a^2)(b^2-v_1^2)(b^2-v_2^2) \\ :&(a^2-b^2)(c^2-v_1^2)(c^2-v_2^2); \end{aligned}$$

therefore $\qquad x_1^2 \qquad : \qquad y_1^2 \qquad : \qquad z_1^2$

$$:: (b^2-c^2)\frac{a^2-v_2^2}{a^2-v_1^2} : (c^2-a^2)\frac{b^2-v_2^2}{b^2-v_1^2} : (a^2-b^2)\frac{c^2-v_2^2}{c^2-v_1^2}.$$

Let α, β, γ be the angles made by the given line of vibration with the elastic axes, then

$$\begin{aligned} (\cos\alpha)^2 &= \frac{x_1^2}{x_1^2+y_1^2+z_1^2} \\ &= (b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2) \end{aligned}$$

divided by

$$(b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2)+(c^2-a^2)(b^2-v_2^2)(c^2-v_1^2)(a^2-v_1^2)$$
$$+(a^2-b^2)(c^2-v_2^2)(a^2-v_1^2)(b^2-v_1^2)$$

and therefore

$$= \frac{(b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2)}{(v_1^2-v_2^2)(a^2-b^2)(b^2-c^2)(c^2-a^2)}$$

(where it is to be observed that the reduction of the denominator is simply the effect of a vast heap of terms disappearing under the influence of contact with the magic circuit (a^2-b^2), (b^2-c^2), (c^2-a^2), a simpler instance of which was seen in Proposition 5).

In fact the coefficient of $v^4 . v^2$

$$\begin{aligned} &= (b^2-c^2)+(c^2-a^2)+(a^2-b^2) \\ &= 0 \end{aligned}$$

that of $v_1^2 . v_2^2$
$$\begin{aligned} &= (c^2+b^2).(c^2-b^2) \\ &\quad + (a^2+c^2).(a^2-c^2) \\ &\quad + (b^2+a^2).(b^2-a^2) \\ &= (c^4-b^4)+(a^4-c^4)+(b^4-a^4) \\ &= 0. \end{aligned}$$

The term in which neither v_1 nor v_2 enters
$$\begin{aligned} &= a^2b^2c^2\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\} \\ &= 0. \end{aligned}$$

The coefficient of
$$-v_1^2 = a^2.(b^4-c^4)+b^2.(c^4-a^4)+c^2.(a^4-b^4)$$
and that of
$$v_2^2 = b^2c^2.(c^2-b^2)+c^2a^2.(a^2-c^2)+a^2b^2.(b^2-a^2)$$
each of which
$$=(a^2-b^2).(b^2-c^2).(c^2-a^2).$$

Hence
$$(\cos\alpha)^2 = \frac{v_1^2-b^2}{v_1^2-v_2^2}.\frac{(a^2-v_2^2)(c^2-v_1^2)}{(a^2-b^2)(a^2-c^2)},$$
in like manner $(\cos\beta)^2 = \&c.$

and
$$(\cos\gamma)^2 = \frac{v_1^2-b^2}{v_1^2-v_2^2}.\frac{(c^2-v_2^2)(a^2-v_1^2)}{(c^2-b^2)(c^2-a^2)}.$$

PROPOSITION 11.

ϵ_1, ϵ_2 being the angles between any line of vibration and the optic axes, required the velocity due to that line in terms of ϵ_1, ϵ_2.

By analytical geometry,
$$\cos\epsilon_1 = \cos\alpha.\cos\phi_1+\cos\gamma.\cos\psi_1$$
$$\cos\epsilon_2 = \cos\alpha.\cos\phi_1-\cos\gamma.\cos\psi_1$$
therefore
$$\cos\epsilon_1.\cos\epsilon_2 = (\cos\alpha)^2(\cos\phi_1)^2-(\cos\gamma)^2(\cos\psi_1)^2$$
$$\begin{aligned} &= \frac{v_1^2-b^2}{v_1^2-v_2^2}.\left\{\frac{(a^2-v_2^2).(c^2-v_1^2)-(c^2-v_2^2).(a^2-v_1^2)}{(a^2-c^2)^2}\right\} \\ &= \frac{v_1^2-b^2}{v_1^2-v_2^2}.\frac{(a^2-c^2)(v_2^2-v_1^2)}{(a^2-c^2)^2} \\ &= \frac{b^2-v_1^2}{a^2-c^2}. \end{aligned}$$

Hence
$$v_1^2 = b^2-(a^2-c^2)\cos\epsilon_1\cos\epsilon_2,$$
and in like manner, for the *conjugate* line of vibration
$$v_2^2 = b^2-(a^2-c^2)\cos\epsilon_1'\cos\epsilon_2'.$$

PROPOSITION 12.

To find ϵ_1, ϵ_2 in terms of ι_1, ι_2.

$$(\cos\epsilon_1)^2+(\cos\epsilon_2)^2=2(\cos\alpha)^2.(\cos\phi_1)^2+2(\cos\gamma)^2.(\cos\psi_1)^2$$

$$=2\frac{v_1^2-b^2}{v_1^2-v_2^2}\left\{\frac{(a^2-v_2^2).(c^2-v_1^2)+(c^2-v_2^2).(a^2-v_1^2)}{(a^2-c^2)^2}\right\}$$

but by Prop. 9

$$v_1^2=a^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2+c^2\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2$$

$$v_2^2=a^2\left(\sin\frac{\iota_1-\iota_2}{2}\right)^2+c^2\left(\cos\frac{\iota_1-\iota_2}{2}\right)^2$$

therefore

$$(\cos\epsilon_1)^2+(\cos\epsilon_2)^2=\frac{b^2-v_1^2}{(a^2-c^2)\sin\iota_1.\sin\iota_2}$$

multiplied by

$$\frac{2(a^2-c^2)^2\left[\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2\left(\sin\frac{\iota_1-\iota_2}{2}\right)^2+\left(\cos\frac{\iota_1-\iota_2}{2}\right)^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2\right]}{(a^2-c^2)^2}$$

$$=\frac{b^2-v_1^2}{(a^2-c^2)\sin\iota_1.\sin\iota_2}\{(\sin\iota_1)^2+(\sin\iota_2)^2\}$$

and we have seen that

$$\cos\epsilon_1\cos\epsilon_2=\frac{b^2-v_1^2}{a^2-c^2}$$

therefore

$$\cos\epsilon_1+\cos\epsilon_2=\sqrt{\left(\frac{b^2-v_1^2}{a^2-c^2}\right)}.\frac{\sin\iota_1+\sin\iota_2}{\sqrt{(\sin\iota_1.\sin\iota_2)}}$$

$$\cos\epsilon_1-\cos\epsilon_2=\sqrt{\left(\frac{b^2-v_1^2}{a^2-c^2}\right)}.\frac{\sin\iota_1-\sin\iota_2}{\sqrt{(\sin\iota_1.\sin\iota_2)}}$$

therefore

$$\cos\epsilon_1=\sqrt{\left\{\frac{b^2-v_1^2}{a^2-c^2}.\frac{\sin\iota_1}{\sin\iota_2}\right\}}$$

$$\cos\epsilon_2=\sqrt{\left\{\frac{b^2-v_1^2}{a^2-c^2}.\frac{\sin\iota_2}{\sin\iota_1}\right\}}$$

and in like manner

$$\cos\epsilon_1'=\sqrt{\left\{\frac{b^2-v_2^2}{a^2-c^2}.\frac{\sin\iota_1}{\sin\iota_2}\right\}}$$

$$\cos\epsilon_2'=\sqrt{\left\{\frac{b^2-v_2^2}{a^2-c^2}.\frac{\sin\iota_2}{\sin\iota_1}\right\}}$$

where v_1, v_2 for the sake of neatness are left *unexpressed* in terms of ι_1, ι_2.

This is the simplest form by which the position of the lines of vibration can be denoted.

Cor. From the last proposition it appears that

$$\frac{\cos \epsilon_1}{\cos \epsilon_2} = \frac{\sin \iota_1}{\sin \iota_2}.$$

Hence we may construct geometrically for the two planes of polarization.

Let I, K be the projections of the two optic axes on a sphere, E the projection of the normal to the front, P the projection of one line of vibration; then

$$\frac{\cos PK}{\cos PI} = \frac{\sin KE}{\sin IE}.$$

Draw FEG the circle of which P is the pole, meeting PK, PI produced in G and F.

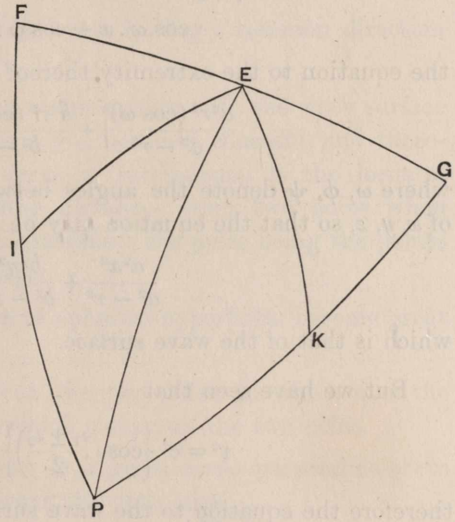

Fig. 2.

Then $\cos PK = \sin KG$,

and $\cos PI = \sin IF$,

therefore

$$\frac{\sin KG}{\sin FI} = \frac{\sin KE}{\sin IE}$$

therefore

$$\frac{\sin KG}{\sin KE} = \frac{\sin IF}{\sin IE}$$

therefore

$$\sin KEG = \sin IEF$$

therefore $KEG = IEF$ or $180° - IEF$. But $PEF = PEG$, therefore EP bisects either the angle IEK or the supplement to it.

These two positions of EP give the two planes of polarization. The construction is the same as that given in Mr Airy's tracts, and originally proposed, I believe, by Mr MacCullagh.

ADDENDUM.

If in the equation of Prop. 6, viz.

$$\frac{(\cos\omega)^2}{a^2-v^2}+\frac{(\cos\phi)^2}{b^2-v^2}+\frac{(\cos\psi)^2}{c^2-v^2}=0$$

we change a, b, c, v into $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{v}$, and consider v to be the length of a line drawn perpendicular to the plane

$$\cos\omega\,.\,x+\cos\phi\,.\,y+\cos\psi\,.\,z=0,$$

the equation to the extremity thereof must be

$$\frac{a^2r^2(\cos\omega)^2}{a^2-r^2}+\frac{b^2r^2(\cos\phi)^2}{b^2-r^2}+\frac{c^2r^2(\cos\psi)^2}{c^2-r^2}$$

where ω, ϕ, ψ denote the angles between the radius vector r, and the axes of x, y, z, so that the equation may be written

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0,$$

which is that of the wave surface.

But we have seen that

$$v^2=c^2\left\{\cos\left(\frac{\iota_1\pm\iota_2}{2}\right)\right\}^2+a^2\left\{\sin\left(\frac{\iota_1\pm\iota_2}{2}\right)\right\}^2,$$

therefore the equation to the wave surface may be written

$$\frac{1}{r^2}=\frac{\left(\cos\frac{\iota_1\pm\iota_2}{2}\right)^2}{c^2}+\frac{\left(\sin\frac{\iota_1\pm\iota_2}{2}\right)^2}{a^2},$$

where ι_1, ι_2 denote the angles between the radius vector v and the two lines which would be the optic axes if a, b, c were changed into $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ so that if e be the inclination of either to the mean axis of elasticity

$$\cos e=\sqrt{\left(\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^2}-\frac{1}{c^2}}\right)}=\frac{c}{b}\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)}$$

$$\sin e=\sqrt{\left(\frac{\frac{1}{b^2}-\frac{1}{c^2}}{\frac{1}{a^2}-\frac{1}{c^2}}\right)}=\frac{a}{b}\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)}.$$

These lines I shall call by way of distinction the prime radii*.

* Upon the authority of Professor Airy I have appropriated the term optic axes to the lines normal to the fronts of single velocity.

COR. 1. If r_1, r_2 be the two values of r corresponding to the same values of ι_1, ι_2 we have

$$\frac{1}{r_1^2}-\frac{1}{r_2^2}=\frac{1}{c^2}\left\{\left(\cos\frac{\iota_1-\iota_2}{2}\right)^2-\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2\right\}$$
$$+\frac{1}{a^2}\left\{\left(\sin\frac{\iota_1-\iota_2}{2}\right)^2-\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2\right\}$$
$$=\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\sin\iota_1 . \sin\iota_2,$$

which proves the celebrated problem of *two rays* having a common direction in a crystal.

COR. 2. The intersection of any concentric sphere with the wave surface is found by making r constant. Hence $\iota_1 \pm \iota_2$ becomes constant, and therefore $r\iota_1 \pm r\iota_2 =$ constant. Hence the curve of intersection is the locus of points, the sum or difference of whose distances from two poles when measured by the arcs of great circles is constant; the poles being the points in which the prime radii pierce the sphere.

In three cases these spherico-ellipses or spherico-hyperbolas become great circles:

(1) When $\iota_1 \pm \iota_2 =$ the angle between the two poles, in which case the curve of intersection is the great circle which comprises the two poles.

(2) When $\iota_1 - \iota_2 = 0$, when the locus is a great circle perpendicular to the former and bisecting the angle between the optic axes.

(3) When $\iota_1 + \iota_2 = 180^\circ$, when the locus is a great circle perpendicular to the two above, and bisecting the supplemental angle between the two axes.

Various other properties may be with the greatest simplicity deduced from the radio-angular equation. The hurry of the press leaves me time only to subjoin the following

PROPOSITION.

To find the inclination of the radius vector to the tangent plane, in terms of the angles which the radius vector makes with the prime radii.

Let O be the centre of the wave surface, OA, OB the two prime radii, OP any radius vector. Let $OP = v$, $POA = \iota_1$, $POB = \iota_2$, and let the inclination of the planes POA, $POB = \mu$;

then $$\frac{1}{r^2}=\frac{\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2}{a^2}+\frac{\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2}{c^2},$$

(taking only the positive sign for the sake of brevity).

Let OQ, OR be the two adjacent radii vectores, so assumed that

$$QOA = POA, \quad QOB = POB + \delta\iota_2,$$

$$ROB = POB, \quad ROA = POA + \delta\iota_1,$$

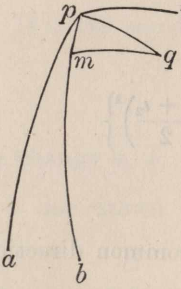

Fig. 3.

and let p, q, r, a, b be the projections of P, Q, R, A, B on a sphere of which O is the centre, then it is clear that

$$qpa = 90°, \quad rpb = 90°,$$

draw qm perpendicular to pb, then $pm = \delta\iota_2$, and therefore

$$pq = \frac{pm}{\sin pqm} = \frac{pm}{\sin apb} = \frac{\delta\iota_2}{\sin \mu}.$$

In like manner

$$pr = \frac{\delta\iota_1}{\sin \mu}.$$

Now the angle QPO

$$= \tan^{-1} . \frac{r . POQ}{OQ - OP} = \tan^{-1} . \frac{r . pq}{\frac{dr}{d\iota_2} . \delta\iota_2};$$

also

$$\frac{d . \frac{1}{r^2}}{d\iota_2} = d\iota_2 \left\{ \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \left(\cos \frac{\iota_1 + \iota_2}{2}\right)^2 \right\}$$

$$= -\left(\frac{1}{c^2} - \frac{1}{a^2}\right) \left(\sin \frac{\iota_1 + \iota_2}{2}\right) \left(\cos \frac{\iota_1 + \iota_2}{2}\right);$$

therefore

$$\frac{dr}{r d\iota_2} = \tfrac{1}{4} r^2 \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin (\iota_1 + \iota_2),$$

therefore

$$\cot QPO = \frac{r^2}{4} \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin (\iota_1 + \iota_2) \sin \mu.$$

In like manner

$$\cot RPO = \frac{r^2}{4} \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin (\iota_1 + \iota_2) \sin \mu,$$

therefore

$$QPO = RPO.$$

Also it is clear that $rpq = apb = \mu$. And to find the inclination of OP to RPQ, we have only to describe a sphere of which P is the centre, and intersecting PQ, PR, PO in Q', R', O'.

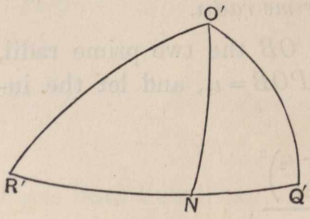

Fig. 4.

Then $R'O'Q' = \mu$, and

$$O'Q' = O'R' = \cot^{-1} \left\{ \frac{r^2}{4} \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin (\iota_1 + \iota_2) \sin \mu \right\}.$$

Draw $O'N$ perpendicular to $R'Q'$, then $O'N$ measures the inclination of the radius vector to the tangent plane*.

And $$Q'O'N = \frac{\mu}{2},$$

therefore $$\cos\frac{\mu}{2} = \tan O'N \,.\, \cot O'Q',$$

therefore $$\cot O'N = \frac{\cot O'Q'}{\cos\frac{\mu}{2}},$$

and therefore
$$\cot O'N = \tfrac{1}{2}r^2 \,.\, \left(\frac{1}{a^2} - \frac{1}{c^2}\right)\sin\frac{\mu}{2} \,.\, \sin(\iota_1 + \iota_2).$$

Let AOB the angle between the optic axes $= 2e$, then by mere trigonometry
$$\sin\frac{\mu}{2} = \sqrt{\frac{\sin\left(e + \frac{\iota_1 - \iota_2}{2}\right)\sin\left(e - \frac{\iota_1 - \iota_2}{2}\right)}{\sin\iota_1 \,.\, \sin\iota_2}},$$
therefore the tangent of the inclination between the radius vector and the normal
$$= \tfrac{1}{2}r^2 \,.\, \left(\frac{1}{a^2} - \frac{1}{c^2}\right)\sin(\iota_1 + \iota_2) \,.\, \sqrt{\frac{\sin\left(e + \frac{\iota_1 - \iota_2}{2}\right)\sin\left(e - \frac{\iota_1 - \iota_2}{2}\right)}{\sin\iota_1 \,.\, \sin\iota_2}}.$$

In like manner the tangent of the inclination between the same radius vector and the normal at the other point of the wave-surface pierced by it
$$= \tfrac{1}{2}(r_1)^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\sin(\iota_1 - \iota_2) \,.\, \sqrt{\frac{\sin\left(e + \frac{\iota_1 + \iota_2}{2}\right)\sin\left(e - \frac{\iota_1 + \iota_2}{2}\right)}{\sin\iota_1 \,.\, \sin\iota_2}}.$$

We may, in the same way, find the inclination of the tangent plane to either of the prime radii, and to the plane which contains them both, in terms of ι_1 and ι_2; the former by a remarkably elegant construction; but the final expressions do not present themselves under the same simple aspect.

If we call ϕ the angle between the ray and the front, we may still further reduce by substituting for r^2 its values in terms of ι_1, ι_2 and we shall obtain
$$\cot\phi = \frac{2(c^2 - a^2)}{c^2\tan\frac{\iota_1 \mp \iota_2}{2} + a^2\cot\frac{\iota_1 \mp \iota_2}{2}}$$
$$\times\sqrt{\left\{\sin\left(e + \frac{\iota_1 \pm \iota_2}{2}\right)\sin\left(e - \frac{\iota_1 \pm \iota_2}{2}\right) \,.\, \operatorname{cosec}\iota_1 \,.\, \operatorname{cosec}\iota_2\right\}}.$$

* O' is the projection of the ray and $R'O'$ of the tangent plane. Therefore $O'N$ being perpendicular to $R'Q'$ represents their inclination.

And if π_1, π_2 be the inclinations of the normal to the two prime radii, it may be shown that

$$\cos \pi_1 = \cos \phi \sin \iota_1 \mp \sin \phi \cos \iota_1 \sin \frac{\mu}{2},$$

$$\cos \pi_2 = \cos \phi \sin \iota_2 \pm \sin \phi \cos \iota_2 \sin \frac{\mu}{2}.$$

Cor. 1. For uniaxal crystals $\frac{\mu}{2} = 90°$ and $\iota_1 + \iota_2 = 180°$, so that the tangent of the inclination of normal to radius vector

$$= \tfrac{1}{2} r^2 . \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin 2\iota \text{ for one point,}$$

and $$= 0 \text{ for the other.}$$

Cor. 2. For every point in the circular section which passes through the poles $\sin \frac{\mu}{2} = 0$, and for the other two circular sections $\iota_1 \pm \iota_2 = 0$ or $180°$.

Therefore every point in the three circular sections is an apse.

Cor. 3. When a nearly $= c$, $\frac{1}{a^2} - \frac{1}{c^2}$ is very small; and therefore the normal and radius vector very nearly coincide.

Cor. 4. Referring to fig. 4 we see that $O'N$ bisects the angle $R'O'Q'$. Now $R'O$, $Q'O$ are respectively perpendicular to the planes passing through O' and the optic axes; and therefore the meridian plane as we may term it, that is, the plane containing both the ray and the normal, always bisects the angle formed by the two planes drawn through the ray and the two optic axes.

Cor. 5. When

$$\iota_1 \text{ or } \iota_2 = 0,$$

$$\iota_2 \text{ or } \iota_1 = e.$$

And therefore ϕ assumes the form $\frac{0}{0}$, which indicates that the extremities of the four prime radii are singular points.

In concluding for the present it behoves me to state that one step has been omitted in the foregoing paper*, viz. the actual performance of the eliminations which lead to the rectilinear equation to the wave-surface. But Mr Archibald Smith's elegant and brief Memoir in the *Cambridge Philosophical Transactions*† of last year leaves nothing to be desired further on that head.

[* See below, p. 27. Ed.] [† Vol. vi. Also *Phil. Mag.* April, 1838, p. 335. Ed.]

That I have not exhibited it in its proper place (Prop. 6) arises only from my respect to the principle of literary propriety. With this important blank supplied the Analytical Theory may be pronounced to be complete.

For all errors and imperfections in what precedes my excuse must be press of time and a total want of the materials to be derived from consulting works of reference.

Since writing the above I have had an opportunity of reading the paper of our living Laplace inserted as part of the Third Supplement to his System of Rays in the *Transactions* of the Royal Irish Academy, in which the principal foregoing results are obtained by aid of a more refined and transcendental analysis.

The nature of the four singular points is there discussed and the existence of four circles of plane contact demonstrated.

The former may be very easily shown thus: when ι_1 is very small $\iota_2 = 2e - \iota_1 \cos\psi$ very nearly, ψ denoting the inclination of the plane in which e is reckoned to the plane in which ι_1 is reckoned.

Hence

$$\left(\frac{1}{r}\right)^2 = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{c^2}\right) - \frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\cos\{2e-\iota_1(\cos\psi \pm 1)\}$$

$$= \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{c^2}\right) - \frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\cos 2e - \frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\sin 2e\,(\cos\psi \pm 1)\,\iota_1$$

$$= \frac{1}{b^2} - \frac{1}{b^2 ac}\sqrt{\{(a^2-b^2)(b^2-c^2)\}}\,(\cos\psi \pm 1)\,\iota_1,$$

therefore

$$r = b\left\{1 + \frac{1}{2}(\cos\psi \pm 1)\left(1-\frac{b^2}{a^2}\right)^{\frac{1}{2}}\left(\frac{b^2}{c^2}-1\right)^{\frac{1}{2}}\iota_1\right\}.$$

Take ψ constant and let the abscissæ and ordinates be reckoned respectively along and perpendicular to the prime ray.

Then $$\iota_1 = \frac{y}{x} \text{ nearly, and } r = \sqrt{(y^2+x^2)} = x,$$

or, if we change the origin to the other extremity of the prime ray,

$$\iota_1 = \frac{y}{b}, \quad r = b - x,$$

so that the equation becomes

$$-\frac{x}{y} = \frac{1}{2}(\cos\psi \pm 1)\sqrt{\left\{\left(1-\frac{b^2}{a^2}\right)\left(\frac{b^2}{c^2}-1\right)\right\}}.$$

Hence at each singular point the surface is touched by a cone, the equation to the generating line of which is given by the above, the extreme angle between it and the prime ray being

$$\cot^{-1}\left[\sqrt{\left\{\left(1-\frac{b^2}{a^2}\right)\left(\frac{b^2}{c^2}-1\right)\right\}}\right].$$

When $b = a$, ψ always $= \frac{\pi}{2}$ and the cone returns into a plane.

Again, let us suppose that the position of any perpendicular from the centre is given, and that of the corresponding radius vector required.

Let OA, OB* denote what we have termed the optic axes, but which it will be more agreeable to analogy to term the prime perpendiculars from centre, and let OP be the given normal. Take OQ, OR contiguous perpendiculars from centre in planes POQ, ROP, perpendicular to POA, POB respectively, then the inclination of the two former will be the same as that of the two latter, and may be termed μ.

Let ι_1, ι_2 now denote the angles POA, POB respectively, then

$$QOA = \iota_1, \qquad QOB = \iota_2 + \delta\iota_2,$$
$$ROA = \iota_1 + \delta\iota_1, \quad ROB = \iota_2.$$

The ray will be found by joining O with the intersection of three planes drawn at P, Q, R, perpendicular to OP, OQ, OR, respectively.

Now from Prop. 9 it appears that

$$OP = \sqrt{\left\{a^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2\right\}},$$

using only one sign for the sake of simplicity, which we may do by throwing the ambiguity upon the *way* in which ι_1 or ι_2 is measured, also

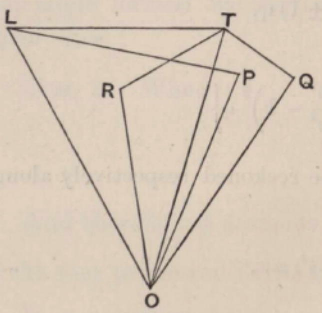

Fig. 5.

$$OQ = OP + \frac{d\,.\,OP}{d\iota_2}\,\delta\iota_2,$$

$$OR = OP + \frac{d\,.\,OP}{d\iota_1}\,\delta\iota_1.$$

Let $\delta\iota_1 = \delta\iota_2$, then it is clear that $OQ = OR$, and the intersection of the two planes perpendicular to OQ, OR is therefore a line perpendicular to the plane QOR, and to the line which bisects the angle QOR.

In fact if we draw QT, RT perpendicular to OQ, OR respectively in the plane QOR, the intersection in question passes through T and is perpendicular to OT; also

$$OT = OQ\,.\,\sec\left(\tfrac{1}{2}ROQ\right) = OQ$$

to the first order of smallness.

* OA, OB are not expressed in the figure.

Now it is easy to see (just as on p. 16) that

$$ROP = \frac{\delta\iota_1}{\sin\mu},$$

and also

$$QOP = \frac{\delta\iota_2}{\sin\mu},$$

therefore $ROP = QOP$ and therefore POT is perpendicular to QOR.

Hence the problem is reduced to finding L the intersection of two lines TL, PL drawn in the *same* plane POT.

Now because OTL, OPL are each right angles, a circle may be made to pass through L, T, P, O.

Hence the angle

$$PLO = PTO = \tan^{-1}\frac{OP \times POT}{OT - OP}$$

$$= \tan^{-1}\frac{OP \times POR \,.\, \cos\tfrac{1}{2}\mu}{\dfrac{d\,.\,OP}{d\iota_2}\,\delta\iota_2} = \tan^{-1}\frac{OP \times \dfrac{\delta\iota_2}{\sin\mu}\cos\tfrac{1}{2}\mu}{\dfrac{d\,.\,OP}{d\iota_2}\,\delta\iota_2},$$

and

$$OL = OP \,.\, \sec POL.$$

Also the position of the plane POL is known, and therefore the radius is completely determined in magnitude and position.

It may be worth while also to remark that the above constructions enable us to form a series of equations between the magnitude of the radius and its inclinations to the two prime perpendiculars.

In fact, if we call π_1, π_2 the two inclinations in question

$$\cos\pi_1 = \cos POL \cos\iota_1 \pm \sin POL \sin\iota_1 \,.\, \sin\frac{\mu}{2},$$

$$\cos\pi_2 = \cos POL \cos\iota_2 \mp \sin POL \sin\iota_2 \,.\, \sin\frac{\mu}{2},$$

and of course if we call the angle between the two prime normals $2E$

$$\sin\frac{\mu}{2} = \sqrt{\frac{\sin\left(E + \frac{\iota_1+\iota_2}{2}\right)\sin\left(E - \frac{\iota_1+\iota_2}{2}\right)}{\sin\iota_1\sin\iota_2}}\,.$$

Cor. 1. When ι_1 or $\iota_2 = 0$, $\tan POL$ assumes the form $\dfrac{0}{0}$ which may be interpreted analogously to the method used in the reverse problem, but may be more elegantly illustrated by

Cor. 2. Which is that the meridian plane POT (that is, the plane in which both normal and radius lie) bisects the angle formed by ROP, QOP, and therefore that formed by the planes drawn through the normal and the two prime normals to which these two are perpendicular.

Now we have found (Cor. 4, page 18), that it also bisects the angle formed by the two planes passing through the radius and the two prime radii. Hence when the ray is given, we may find by the easiest geometry the normal and the tangent plane, and *vice versâ*.

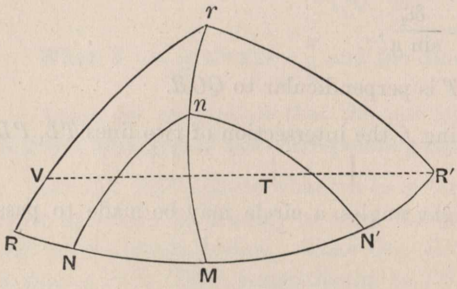

Fig. 6.

Thus suppose (N, N') (R, R') to be the projections of the prime perpendiculars and prime radii on a sphere concentric with the wave surface.

Let n be the projection of any given perpendicular on the same sphere; join nN, nN'; bisect NnN' by nM, which will be the meridian plane.

Draw from R', $R'TV$ perpendicular to nM and make $R'T = TV$. Produce RV to meet Mn in r, then $RrM = R'rM$, and therefore r is the projection of the radius. Just in the same way when r is given we may find n.

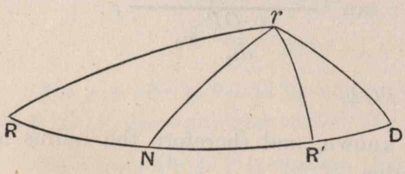

Fig. 7.

Now suppose n to come to N, then the position of the meridian plane nM becomes indeterminate, and r from a point becomes a locus, subject to the condition that $R'rN = RrN$. From r draw rD perpendicular to rN.

Then it is clear that because rN bisects RrR'

$$\frac{\sin RD}{\sin R'D} = \frac{\sin Rr}{\sin R'r} = \frac{\sin RN}{\sin R'N},$$

and therefore D is a fixed point and ND a fixed length, and

$$\cos rND = \tan rN \,.\, \cot ND\,;$$

therefore the projection of the locus of r upon a plane drawn at N perpendicular to the line joining N with the centre O is given by the equation

$$\rho = ON \,.\, \cot ND \,.\, \cos\theta,$$

N being the origin and the projection of ND the prime radius; which is the equation to a circle passing through N, and whose diameter $= ON \cot ND$.

Hence at the extremity of each prime perpendicular the tangent plane meets the surface in a circle passing through that extremity and whose radius $= \frac{1}{2}b \cot\alpha$, α being to be found from the equation

$$\frac{\sin(2E+\alpha)}{\sin\alpha} = \frac{\sin(E+e)}{\sin(E-e)},$$

that is $$\tan(E+\alpha) = (\tan E)^2 \cot e.$$

Just in the same way it may be shown that the trace of the perpendiculars to the tangent planes of the surface at the point where it is pierced by any prime radius upon a plane perpendicular to that radius at its extremity, is also a circle passing through it, and curved in an opposite direction from the circle of plane contact nearest to it.

Hence the enveloping cone at these points may be described as being perpendicular to the circular cone, formed by drawing lines from the centre to the above described circle; that is every generating line of the one will be perpendicular to the generating line which it meets of the other.

More generally it easily appears from fig. 6 that if a series of great circles (representing meridian planes) be taken intersecting the great circle $NRR'N'$ in a fixed point, a plane perpendicular to the radius passing through that point, will intersect the cone of rays as well as the cone of perpendiculars corresponding to those meridian planes, in two *circles*. So that there exist an indefinite number of *circular* cones of rays corresponding to *circular* cones of perpendiculars touching each other in a line lying in the plane containing the extreme axes, and having their circular sections perpendicular to that line.

The cusps are explained by the cone of *rays* degenerating into a right line, and the circles of plane contact by the cone of perpendiculars so degenerating.

Furthermore I observe in conclusion that when a ray is given it follows from the general geometrical construction above that there will be two meridian planes according as we take R with R', or with a point 180° from R', and consequently these two planes will be perpendicular to each other.

And similarly when a *normal* is given there will be two meridian planes perpendicular to each other.

Thus the planes passing through any radius and the two normals at the points where it pierces the wave surface, are *perpendicular* to each other, as are also the two planes passing through any normal and *its* two corresponding radii.

Moreover a glance at fig. 2 will show that the two lines of vibration corresponding to any front lie respectively in the two meridian planes passing through the perpendicular to that front or, in other words, the intersection of a plane drawn through either ray belonging to a front perpendicular thereunto is *always* a line of vibration in that front.

This has been noticed, I think, by Sir William Hamilton for the particular case of the singular points.

As two fronts belong to every ray, so two rays pertain to every front. And from what has been said above it appears that the two lines of vibration in any front are the projections of its two rays upon its own plane.

NOTE 1.

In the paper above, it is shown that the meridian plane, that is, the plane containing the ray and normal, always passes through a line of vibration in the corresponding point. Now the line of force called into action by a displacement in the line of vibration clearly lies in this very plane; for the resolved part of it lies in the line of vibration itself.

Harmony and analogy concur in suggesting that as two of these four lines are perpendicular to each other, so are also the other two, or in other words, that the ray is always perpendicular to the direction of unresolved force.

The following investigation verifies this conjecture.

Let x, y, z be the coordinates of a point taken at distance unity from the origin and in any line of vibration; then the cosines of the angles made by the line of force with the axes are as $a^2x : b^2y : c^2z$ respectively.

Let ω be the inclination between the line of vibration and the line of force, then

$$\cos\omega = \frac{a^2x \,.\, x + b^2y \,.\, y + c^2z \,.\, z}{\sqrt{\{(a^4x^2 + b^4y^2 + c^4z^2)(x^2 + y^2 + z^2)\}}} = \frac{a^2x^2 + b^2y^2 + c^2z^2}{\sqrt{(a^4x^2 + b^4y^2 + c^4z^2)}}.$$

Let $$\sqrt{(a^4x^2 + b^4y^2 + c^4z^2)} = P,$$

then $$P^2 = v^4 (\sec\omega)^2.$$

Now let α, β, γ be the angles of inclination between the coordinate planes and the front in which the line of vibration lies, and λ some quantity to be determined. I have shown in Prop. 3 that if

$$\lambda\cos\alpha = (a^2 - v^2)\, x,$$

then will $$\lambda\cos\beta = (b^2 - v^2)\, y,$$

and $$\lambda\cos\gamma = (c^2 - v^2)\, z;$$

therefore $$\lambda^2 = a^4x^2 + b^4y^2 + c^4z^2 - 2v^2(a^2x^2 + b^2y^2 + c^2z^2) + v^4 = P^2 - v^4.$$

Again,

$$\lambda^2 \,.\, \left(\frac{(\cos\alpha)^2}{(a^2 - v^2)^2} + \frac{(\cos\beta)^2}{(b^2 - v^2)^2} + \frac{(\cos\gamma)^2}{(c^2 - v^2)^2}\right) = x^2 + y^2 + z^2 = 1;$$

therefore $$\frac{1}{P^2 - v^4} = \frac{(\cos\alpha)^2}{(a^2 - v^2)^2} + \frac{(\cos\beta)^2}{(b^2 - v^2)^2} + \frac{(\cos\gamma)^2}{(c^2 - v^2)^2}.$$

Now $$\frac{1}{P^2 - v^4} = \frac{1}{v^4(\sec\omega)^2 - v^4} = \frac{1}{v^4}(\cot\omega)^2.$$

And in Mr Smith's investigation of the form of the wave surface (already alluded to*) by great good fortune I find ready to my hand

$$\frac{(\cos\alpha)^2}{(a^2-v^2)^2}+\frac{(\cos\beta)^2}{(b^2-v^2)^2}+\frac{(\cos\gamma)^2}{(c^2-v^2)^2}=\frac{1}{v^2(r^2-v^2)},$$

r being the radius vector to the point whose tangent plane is parallel to the point in question.

Hence

$$(\cot\omega)^2=\frac{v^4}{v^2(r^2-v^2)}=\frac{v^2}{r^2-v^2}=\frac{p^2}{r^2-p^2},$$

p being the length of the perpendicular from the centre upon the tangent plane, for $p=v$.

Hence $(\cot\omega)^2=$ the square of the cotangent of the angle between radius vector and normal.

Or, in other words, the line of force is as much inclined to the line of vibration as the ray is to the normal.

Now the normal is perpendicular to the line of vibration, and all four lines lie in one plane.

Therefore the ray is perpendicular to the line of force. Q. E. D.

I may be allowed to conclude this long paper with a summary of some of the most remarkable consequences which I have extricated from Fresnel's hypothesis.

(1) The two meridian planes corresponding to any given radius are perpendicular to each other†.

(2) So are the two corresponding to any given normal.

(3) Every meridian plane bisects the angle formed by two planes drawn through the radius and the two prime radii.

(4) It also bisects the angle formed by two planes drawn through the normal and the two prime normals.

(5) Each meridian plane contains one line of vibration and the corresponding line of force.

(6) The ray is perpendicular to the line of force.

All these conclusions, except the fourth, are, I believe, original.

* See above, p. 18.

† I have defined the meridian plane to be that which contains radius vector and normal belonging to the same point.

The theory of external and internal conical refraction follows immediately as a particular consequence from the third and fourth combined as already shown; the same propositions also enable us to draw a tangent plane to any point of the wave surface by mere Euclidean geometry. May not some of these conclusions serve to suggest to physical inquirers the question, Has the theory been started from the most natural point of view?

NOTE 2. *Investigation* of the Wave Surface.*

Since the appearance of the preceding parts, I have succeeded in completing the self-sufficiency of my method by deducing the equation to the wave surface from the expressions given in Prop. 5 for the angles between a front and the principal planes in terms of its two velocities. If these angles be ω, ϕ, ψ, and the two velocities v_1, v_2 we found

$$\cos\omega = \sqrt{\frac{(a^2 - v_1^2)(a^2 - v_2^2)}{(a^2 - b^2)(a^2 - c^2)}},$$

$$\cos\phi = \sqrt{\frac{(b^2 - v_1^2)(b^2 - v_2^2)}{(b^2 - a^2)(b^2 - c^2)}},$$

$$\cos\psi = \sqrt{\frac{(c^2 - v_1^2)(c^2 - v_2^2)}{(c^2 - a^2)(c^2 - b^2)}}.$$

Let the tangent plane to the wave surface be written

$$\frac{\cos\omega}{v_1}\,.\,x + \frac{\cos\phi}{v_1}\,.\,y + \frac{\cos\psi}{v_1}\,.\,z = 1, \qquad (\alpha)\dagger$$

then

$$\frac{d\,\dfrac{\cos\omega}{v_1}\ddagger}{d\left(\dfrac{1}{v_1}\right)^2}x + \frac{d\,\dfrac{\cos\phi}{v_1}}{d\left(\dfrac{1}{v_1}\right)^2}y + \frac{d\,\dfrac{\cos\psi}{v_1}}{d\left(\dfrac{1}{v_1}\right)^2}z = 0, \qquad (\beta)$$

$$\frac{d\cos\omega}{d\,(v_2)^2}x + \frac{d\cos\phi}{d\,(v_2)^2}y + \frac{d\cos\psi}{d\,(v_2)^2}z = 0. \qquad (\gamma)$$

Let

$$\frac{1}{v_1}\sqrt{\frac{(a^2 - v_1^2)}{(a^2 - v_2^2)}} = \xi, \qquad \sqrt{\{(a^2 - b^2)(a^2 - c^2)\}} = \frac{1}{A},$$

$$\frac{1}{v_1}\sqrt{\frac{(b^2 - v_1^2)}{(b^2 - v_2^2)}} = \eta, \qquad \sqrt{\{(b^2 - a^2)(b^2 - c^2)\}} = \frac{1}{B},$$

$$\frac{1}{v_1}\sqrt{\frac{(c^2 - v_1^2)}{(c^2 - v_2^2)}} = \zeta, \qquad \sqrt{\{(c^2 - a^2)(c^2 - b^2)\}} = \frac{1}{C},$$

* This investigation supplies the step which Mr Tovey was desirous should appear in the *Magazine*. [*Phil. Mag.* March, 1838, p. 261. ED.]

† In lieu of v_1 we might write v_2 in the denominator without affecting the result.

‡ Observe, that $\dfrac{\cos\omega}{v_1} = \dfrac{\sqrt{\left\{\left(\dfrac{a^2}{v_1^2} - 1\right)(a^2 - v_2^2)\right\}}}{\sqrt{\{(a^2 - b^2)(a^2 - c^2)\}}}$, and so on for the rest.

then equation (γ) becomes

$$A\xi x + B\eta y + C\zeta z = 0, \qquad (1)$$

and equation (β)

$$\frac{Aa^2}{\xi}x + \frac{Bb^2}{\eta}y + \frac{Cc^2}{\zeta}z = 0, \qquad (2)$$

and equation (α) may be written under two forms, viz.

$$(a^2 - v_2{}^2)\, A\xi x + (b^2 - v_2{}^2)\, B\eta y + (c^2 - v_2{}^2)\, C\zeta z = 1, \qquad (3)$$

or

$$\left(\frac{a^2}{v_1{}^2} - 1\right)\frac{A}{\xi}x + \left(\frac{b^2}{v_1{}^2} - 1\right)\frac{B}{\eta}y + \left(\frac{c^2}{v_1{}^2} - 1\right)\frac{C}{\zeta}z = 1. \qquad (4)$$

From (1)

$$A\xi x + B\eta y = -C\zeta z. \qquad (5)$$

From (2)

$$\frac{Aa^2}{\xi}x + \frac{Bb^2}{\eta}y = -\frac{Cc^2}{\zeta}z. \qquad (6)$$

From (3) and (1)

$$A\,(a^2 - c^2)\,\xi x + B\,(b^2 - c^2)\,\eta y = 1. \qquad (7)$$

From (2) and (4)

$$A\,(a^2 - c^2)\frac{x}{\xi} + B\,(b^2 - c^2)\frac{y}{\eta} = c^2. \qquad (8)$$

From (5) and (6)

$$C^2c^2z^2 - B^2b^2y^2 - A^2a^2x^2 = ABxy\left(a^2\frac{\eta}{\xi} + b^2\frac{\xi}{\eta}\right). \qquad (9)$$

From (7) and (8)

$$c^2 - B^2(b^2 - c^2)^2y^2 - A^2(a^2 - c^2)^2x^2 = ABxy\left(\frac{\eta}{\xi} + \frac{\xi}{\eta}\right) \times (a^2 - c^2)(b^2 - c^2). \qquad (10)$$

From (9) and (10)

$$AB\,(a^2 - b^2)\,(a^2 - c^2)\,(b^2 - c^2)\,xy\frac{\xi}{\eta} = a^2c^2 - (a^2 - c^2)\,(b^2 - c^2)\,C^2c^2z^2$$
$$- \{a^2(b^2 - c^2)^2 - b^2(a^2 - c^2)\,(b^2 - c^2)\}\,B^2y^2$$
$$- \{a^2(a^2 - c^2)^2 - a^2(a^2 - c^2)\,(b^2 - c^2)\}\,A^2x^2 = a^2c^2 - c^2z^2 - c^2y^2 - a^2x^2. \qquad (11)$$

From (11), interchanging (a, x, ξ) with (b, y, η) we have

$$AB\,(b^2 - a^2)\,(b^2 - c^2)\,(a^2 - c^2)\,xy\frac{\eta}{\xi} = b^2c^2 - c^2z^2 - c^2x^2 - b^2y^2. \qquad (12)$$

Finally, from (11) and (12) we have

$$\{a^2c^2 - (a^2 - c^2)\,x^2 - c^2(x^2 + y^2 + z^2)\}\,\{b^2c^2 - (b^2 - c^2)\,y^2 - c^2(x^2 + y^2 + z^2)\}$$
$$= (a^2 - c^2)\,(b^2 - c^2)\,x^2y^2,$$

that is

$$(x^2 + y^2 + z^2)\,(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)\,x^2$$
$$- b^2(a^2 + c^2)\,y^2 - c^2(b^2 + a^2)\,z^2 + a^2b^2c^2 = 0$$

the equation required.