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ON DEFINITE DOUBLE INTEGRATION, SUPPLEMENTARY TO A FORMER PAPER ON THE MOTION AND REST OF FLUIDS.

[*Philosophical Magazine*, XIV. (1839), pp. 298—300.]

In a paper on Fluids which appeared in the December Number of this *Magazine*, I had occasion to remark, that the mass of an area having at the point (x, y) a density $\frac{du}{dx}+\frac{dv}{dy}$ could be expressed by the simple formula

$$\int_l^0 \left\{u\frac{dy}{ds}-v\frac{dx}{ds}\right\} ds\,;$$

l being the length, and ds an element of the bounding curve: this may be thought to require some explanation.

(1) Let $APBq$ represent any oval; PpL, QqM any two contiguous ordinates cutting the curve in Pp, Qq respectively, AC, BD the two extreme tangents parallel to Oy, and ρ the density at any point (x, y). The expression $\iint\rho dxdy$ will serve to denote the mass of the oval area $APBq$, and the limits may be twice taken, that is, (i) the two values of y corresponding to any one of x; and (ii) the two values of x corresponding to C and D. This method is in fact tantamount to taking the sum of the columns Pp qQ; but this is not necessary, for

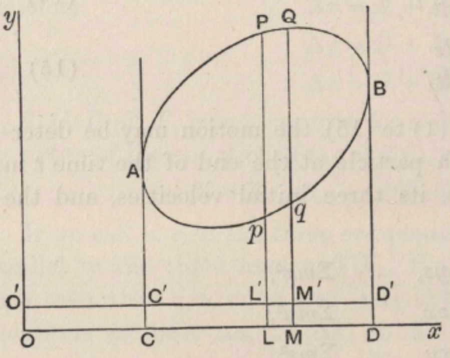

Fig. 1.

$APBq$ may be considered as the *algebraical* sum of the mixtilinear area $APQBDC$, and the mixtilinear area $BDCApq$, or (if any line $O'C'D'$ be drawn parallel to $OCLMD$) of $APQBD'C'$ and $BD'C'Apq$.

Thus then the mass $=\int dx\,(\int\rho dy)$, $\int\rho dy$ being left indeterminate, and the extremity of x travelled round from C to D, and back again from D to C.

This will be better expressed by transforming the variable, and summing with respect to some quantity, such as the arc of the curve, which continuously increases, or if we please, with respect to θ, the angle subtending any point taken within the curve.

The mass is then

$$= \pm \int_{2\pi}^{0} d\theta \left\{ (\int \rho dy) \frac{dx}{d\theta} \right\};$$

always remembering that *no* constant need be added to $\int \rho dy$, and that the doubtful sign arises from the choice of ways in which θ may be measured round. If the area be not included by one line; but by several, as for example, by a curve and a right line, the above integral, if broken up into as many parts as there are breaches of continuity, will still apply.

(2) Let us suppose that we have two areas exactly coinciding with, and overlapping one another; but the density of the one at (x, y) to be ρ, and of the other ρ'.

Let the mass of the first be treated as the sum of columns parallel to Oy, and that of the second as the sum of columns parallel to Ox.

The one will be represented by

$$\pm \int_{2\pi}^{0} d\theta \, (\int \rho dy) \frac{dx}{d\theta},$$

the other will be represented by

$$\pm \int_{2\pi}^{0} d\theta \, (\int \rho' dx) \frac{dy}{d\theta},$$

and the sum of the two, or the joint mass, by

$$\pm \int_{2\pi}^{0} d\theta \, \{\int \rho dy\} \frac{dx}{d\theta} \pm \int_{2\pi}^{0} d\theta \, \{\int \rho' dx\} \frac{dy}{d\theta}.$$

So long as these two operations are performed separately, the doubtful signs may be preserved in each term, because s need not be travelled round in the same direction for the two summations; but if we perform the second integration conjointly for the two masses, their sum

$$= \pm \int_{2\pi}^{0} d\theta \left\{ (\int \rho dy) \frac{dx}{d\theta} \pm ? \, (\int \rho' dx) \frac{dy}{d\theta} \right\},$$

the mark of interrogation denoting that *one or the other*, but not *either* of the signs $\pm$ must be used, and the question is, which ?

This will be answered by taking different points in the bounding line which may be continuous or not. Now every line returning into itself, whether continuous or not, will naturally divide with respect of any given system of axes, into at most four parts, or sets of parts; two in which dx and dy both increase or both decrease, and two in which one increases and the other decreases.

Take P_1, P_2, P_3, P_4, any points in the four quadrants respectively, it will be observed that,

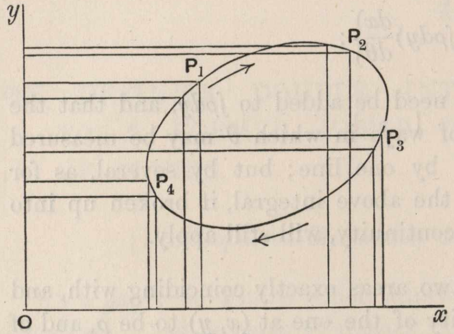

Fig. 2.

At P_1 the ρ column enters additively, and the ρ' column subtractively.

At P_2 both columns are additive.

At P_3 the ρ' column is additive and the ρ column subtractive.

At P_4 both columns enter subtractively.

Again, reckoning round in the direction of the arrows,

At P_1, x and y are both increasing.

At P_2, x is increasing and y decreasing.

At P_3, x and y both decrease.

At P_4, x is decreasing and y increasing.

Thus when $\int\rho dy$ and $\int\rho' dx$ are affected with the same signs, dx and dy are of opposite signs; and when $\int\rho dy$, $\int\rho' dx$ are of opposite signs, dx and dy are of the same sign.

Hence it appears that the mass of the area, whose density at (x, y) is $\rho+\rho'$, is capable of being represented by

$$\pm\int_{2\pi}^{0} d\theta\left\{(\int\rho dy)\frac{dx}{d\theta}-(\int\rho' dx)\frac{dy}{d\theta}\right\}.$$