7.

ON RATIONAL DERIVATION FROM EQUATIONS OF COEXISTENCE, THAT IS TO SAY, A NEW AND EXTENDED THEORY OF ELIMINATION*. PART I.

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ANY number of equations existing at the same time and having the same quantities repeated, may be termed equations of coexistence: in the *present* paper we consider only the case of *two* algebraical equations:

$$x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots\ldots + a_m = 0,$$

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots\ldots + b_n = 0.$$

The above being "equations of coexistence," x is called "the repeating term."

If we suppose the equation

$$c_0x^r + c_1x^{r-1} + c_2x^{r-2} + \ldots\ldots + c_r = 0$$

to be capable of being deduced from the two above, and, therefore, necessarily implied by them, this will be called "a Particular Derivative" from the equations of coexistence, of the rth degree, (r being supposed less than m and n†, and the coefficients being *rational* functions of the coefficients of the equations of coexistence).

There will be an *indefinite* number in general of such derivatives, and the form involving arbitrary quantities which includes them *all* is called "the general derivative of the rth degree."

Any "Particular Derivative," in which the terms are all integral, numerically as well as literally speaking, is called an "Integral Derivative."

That "Integral Derivative" of any given degree in which the literal parts of the coefficients are of the *lowest possible dimensions*‡, and the numerical parts as low as they can be made, is called the "Prime Derivative"

[* The results of this and some following papers were repeated, with demonstrations, in the paper "On a Theory of the Syzygetic Relations of two rational integral functions comprising an application to the Theory of Sturm's Functions, and that of the greatest Algebraical Common Measure," *Phil. Trans. Royal Soc.* Vol. CXLIII., Part I. pp. 407—548, 1853. See below Section II. Art. (16) of that paper. ED.]

† This restriction upon the value of r is not essentially requisite, and is only introduced to keep the attention fixed upon the particular objects of this first Part.

‡ Of course the dimensions of the coefficients in the equations of coexistence are to be understood as denoted by the indices subscribed.

of that degree. So that there is nothing left ambiguous in the prime derivative save the sign.

The "Derivative by succession" is that particular derivative which is obtained by performing upon the equations of coexistence the process commonly employed for the discovery of the greatest common measure, and equating the *successive remainders* to zero.

To express the product of the sums formed by adding each of one row of quantities to each of another row, we simply write the one row above the other; a notation clearly capable of extension to any number of rows, which would not be the case if we spoke of *differences* instead of *sums**.

Theorem 1.

Let $h_1, h_2, \ldots h_m$, be the roots of one equation of coexistence, $k_1, k_2, \ldots k_n$, the roots of the other. The general derivative of the rth degree is represented by

$$\Sigma\left(SR(h_1, h_2, h_3 \ldots h_r)\{(x-h_1)(x-h_2)\ldots(x-h_r)\} \times \begin{Bmatrix} h_{r+1}, h_{r+2} \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix}\right) = 0,$$

$SR(h_1, h_2, h_3 \ldots h_r)$ denoting any *symmetrical rational* (integral or fractional) function of $h_1, h_2 \ldots h_r$;

$$\begin{Bmatrix} h_{r+1}, h_{r+2} \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix}$$

being to be interpreted as above explained, and Σ of course including as many terms as there are ways of putting n things r and r together†.

A form tantamount to the above, and which may be substituted for it, is its analogue,

$$\Sigma\left(SR(k_1, k_2 \ldots k_r)\{(x-k_1)(x-k_2)\ldots(x-k_r)\} \times \begin{Bmatrix} k_{r+1}, k_{r+2} \ldots k_n \\ -h_1, -h_2 \ldots -h_m \end{Bmatrix}\right) = 0.$$

When $r = 0$ the theorem gives simply

$$\begin{Bmatrix} h_1, h_2 \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix} = 0,$$

and is coincident with that given by Bezout in his Theory of *Elimination*.

* The wider views which I have attained since writing the above, and which will be developed in a future paper, lead me to request that this notation may be considered only as temporary. It would have been more in accordance with these views to have used the two rows to denote products of differences than of sums. But a change now in the text would be very apt to cause errors in printing.

† The general derivative may clearly be expressed also by the sum of any two particular derivatives affected respectively with arbitrary rational coefficients. The equivalency of an arbitrary *function* to two arbitrary *multipliers* is very remarkable, and analogous to what occurs in the solution of certain differential equations.

Subsidiary Theorem (A).

If $h_1, h_2 \dots h_m$ be the roots of the equation

$$x^m + a_1 x^{m-1} + a_2 x^{m-2} + \dots\dots + a_m = 0,$$

and if
$$e^m + a_1 e^{m-1} + a_2 e^{m-2} + \dots\dots + a_m - u = 0,$$

then
$$\Sigma \frac{h_1{}^r}{(h_1 - h_2)(h_1 - h_3) \dots (h_1 - h_m)} = \frac{1}{r+1} \frac{d}{du} \Sigma (e^{r+1}),$$

u being made zero after differentiation.

Cor. If $R(h_1)$ denote any integral rational function of h_1, then

$$\Sigma \frac{R(h_1)}{(h_1 - h_2)(h_1 - h_3) \dots (h_1 - h_m)}$$

is *always* integral and is *zero* when the dimensions of $R(h_1)$ fall short of $(m-1)$.

Subsidiary Theorem (B).

$$\Sigma \frac{SR(h_1, h_2 \dots h_r)}{\begin{Bmatrix} h_1, h_2 \dots h_r \\ -h_{r+1}, -h_{r+2} \dots -h_m \end{Bmatrix}}$$

can be expressed by the sum of terms, each of which is the product of series of the form

$$\Sigma \frac{R(h_1)}{(h_1 - h_2)(h_1 - h_3) \dots (h_1 - h_m)},$$

it is *always* integral, and when the dimensions of the numerator fall short of $(m-r)r$ it vanishes*.

Subsidiary Theorem (C).

The *only* modes of satisfying the equation

$$\Sigma \{f(h_1, h_2 \dots h_r) \times SR(h_1, h_2 \dots h_r)\} = 0,$$

for all forms of the latter factors short of $(m-r)(n-r)$ dimensions, are to put $f(h_1, h_2 \dots h_r) = 0$, or else

$$f(h_1, h_2 \dots h_r) = \frac{\text{constant}}{\begin{pmatrix} h_1, h_2 \dots h_r \\ -h_{r+1}, -h_{r+2} \dots -h_m \end{pmatrix}}.$$

* It may be remarked also in passing, that any term in the numerator which contains *any* one power not greater than $m - 2r$ may be neglected and thrown out of calculation. Moreover, an analogous proposition may be stated of fractions in the denominators of which *any number* of rows are written one under the other; see the first note, page 41.

THEOREM 2.

By virtue of the subsidiary theorem (B), the two equations

$$\pm \Sigma \left((x-h_1)(x-h_2)\dots(x-h_r) \times \frac{\begin{Bmatrix} h_{r+1}, \ h_{r+2} \dots h_m \\ -k_1, \ -k_2 \dots -k_n \end{Bmatrix}}{\begin{Bmatrix} h_{r+1}, \ h_{r+2} \dots h_m \\ -h_1, \ -h_2 \dots -h_r \end{Bmatrix}} \right) = 0,$$

$$\pm \Sigma \left((x-k_1)(x-k_2)\dots(x-k_r) \times \frac{\begin{Bmatrix} k_{r+1}, \ k_{r+2} \dots k_n \\ -h_1, \ -h_2 \dots -h_m \end{Bmatrix}}{\begin{Bmatrix} k_{r+1}, \ k_{r+2} \dots k_n \\ -k_1, \ -k_2 \dots -k_r \end{Bmatrix}} \right) = 0,$$

are each integer derivatives of the rth degree.

THEOREM 3.

And by virtue of the subsidiary theorem (C), the two above equations are the "Prime Integer Derivatives," and are exactly identical with each other.

COR. 1. The leading coefficient of the "prime derivative" of the rth degree is always of $(m-r)(n-r)$ dimensions.

COR. 2. If P_r be the prime derivative of the rth degree and if $(X=0, Y=0)$ be the two equations of coexistence, and λ_r, μ_r the two "prime constituents of multiplication" to the said derivative, that is if λ_r and μ_r satisfy the equation $\lambda_r X + \mu_r Y = P_r$, then the coefficient of the leading terms in λ_r and in μ_r is of $(m-r-1)(n-r-1)$ dimensions.

THEOREM 4.

The "Prime Derivative" of any given degree is an exact factor of the "derivative by succession," of the same degree. The quotient resulting from striking out this factor is called "the quotient of succession."

THEOREM 5.

If L_1, L_2, L_3, &c., be the leading coefficients of the derivatives occurring first, second, third, &c., in order after the equations of coexistence, and if Q_1, Q_2, Q_3, &c., represent the first, second, third, "quotients of succession" reckoned in the same order, then

$$Q_1 = 1,$$

$$Q_2 = \frac{1}{L_1^2},$$

$$Q_3 = \frac{L_1^4}{L_2^2},$$

$$Q_4 = \frac{L_2^4}{L_1^4 L_3^2},$$

and in general

$$Q_{2n} = \frac{L_2^4 L_4^4 \ldots L_{2n-4}^4 L_{2n-2}^4}{L_1^4 L_3^4 \ldots L_{2n-3}^4 L_{2n-1}^2} * ,$$

$$Q_{2n+1} = \frac{L_1^4 L_3^4 \ldots L_{2n-3}^4 L_{2n-1}^4}{L_2^4 L_4^4 \ldots L_{2n-2}^4 L_{2n}^2} \dagger.$$

COR. Hence, in place of Sturm's auxiliary functions, we may substitute the functions derived from the equations of coexistence $\left(fx = 0, \frac{dfx}{dx} = 0\right)$ according to Theorem 2, due regard being had to the sign.

Scholium. Hitherto it has been supposed that the values of the coefficients in the equations of coexistence are independent of one another, but particular relations may be supposed to exist which shall cause the leading terms given by Theorem 2 to vanish, giving rise to anormal or singular primes, as they may be called, of the degree r of fewer than $(m-r)(n-r)$ dimensions. The theory of this, the failing case (so to say), is highly interesting, and I have already discovered the law of formation for the quotients of succession on the supposition of *any number* of primes vanishing consecutively; but I forbear to vex the patience of my reader further, the more so, as I hope soon to be able to present a complete memoir, with all the steps here indicated filled up, and numerous important additions, (the perfect image of which this is but a rough mould), as homage to the learned and illustrious society which has lately done me the honour of admitting me into its ranks.

Why this has not already been done must be excused, by the fact of the theory having suggested itself abroad in the intervals of sickness‡. Yet thus much will I add in general terms, namely, that as many primes as vanish consecutively, so many units must be added to the index 2 of the accessions

* That the appearance of the index 4 may not startle, let my reader bear in mind that there are what may be termed secondary derivatives of succession for every degree appearing in the process of successive division.

† The prime derivatives must be capable of yielding an *internal* evidence of the truth of Sturm's theorem. In fact, for the case of all the roots being possible, a little consideration will serve to show that the leading term of each prime derivative of the equation $\left\{fx \frac{dfx}{dx}\right\} = 0$ will consist of a series of fractions, each of which fractions is, *numerically* speaking, of the *same sign*.

‡ The reflections which Sturm's memorable theorem had originally excited, were revived by happening to be present at a sitting of the French Institute, where a letter was read from the Minister of Public Instruction, requesting an opinion upon the expediency of forming tables of elimination between two equations as high as the 5th or 6th degree containing one repeating term. The offer was rejected, on the ground of the excessive labour that would be required. I think that this has been very much overrated; and probably many will be of the same opinion who have dwelt upon the fact that no numerical quantity will occur in the result higher than the highest index of the repeating term. Would it not redound to the honour of British science that some painstaking ingenious person should gird himself to the task? and would not this be a proper object to meet with encouragement from the Scientific Association of Great Britain?

received in the numerator and denominator of the subsequent quotient; and in the quotient after that, it is not the square of the leading term of the penultimate prime,—but the product of this term by the leading term of that anormal prime of the same degree which has the lowest dimensions,—that finds its way into the numerator. The rest of the formation remaining undisturbed, *unless* and *until* a new failure have taken place.

Note on Sturm's Theorem.

When one of the equations of coexistence is the differential coefficient with respect to the repeated term of the other, the prime derivatives given in Theorem 2 which coincide in this case with Sturm's auxiliary functions *reduced* to their lowest terms, may be exhibited under an integral *aspect.*

Let SPD intimate that the squared product of the differences is to be taken of the quantities which follow it.

Let S_1 indicate the sum of the quantities to which it is prefixed.

S_2 the sum of the binary products.

S_3 the sum of the ternary products, and so on

Let $h_1, h_2 \dots h_n$ be the roots of any equation.

Then Sturm's last auxiliary function may be *replaced* by

$$SPD\,(h_1, h_2 \dots h_n).$$

The last but one may be replaced by

$$\Sigma SPD\,(h_1, h_2 \dots h_{n-1})\,x + \Sigma S_1\,(h_2, h_3 \dots h_{n-1})\,SPD\,(h_1, h_2 \dots h_{n-1}).$$

The one preceding by

$$\Sigma SPD\,(h_1, h_2 \dots h_{n-2})\,x^2 + \Sigma S_1\,(h_1, h_2 \dots h_{n-2})\,SPD\,(h_1, h_2 \dots h_{n-2})\,x$$
$$+ \Sigma S_2\,(h_1, h_2 \dots h_{n-2})\,SPD\,(h_1, h_2 \dots h_{n-2}),$$

and so on.

Thus then Sturm's rule for determining the absolute number of real roots in an equation is based wholly and solely upon the following

Algebraical Proposition.

If there be n quantities, real and imaginary, the imaginary ones entering in pairs, as many changes of sign as there are in the terms

$$\Sigma SPD\,(h_1, h_2),$$
$$\Sigma SPD\,(h_1, h_2, h_3),$$
$$\dots\dots\dots\dots\dots\dots$$
$$\Sigma SPD\,(h_1, h_2 \dots h_{n-1}),$$
$$\Sigma SPD\,(h_1, h_2 \dots h_n),$$

so many in number are these pairs.

Query (1). Is there no proposition applicable to any n quantities *whatever*?

Query (2). Is there no faintly analogous proposition applicable to higher powers than the squares?

Query (3). Seeing that in forming the coefficients in the equation of the squares of the differences, we pass from n functions of the roots to $n\frac{n-1}{2}$ and not n functions, of their squared differences, does not a natural passage to the former lie through n functions of the squared differences?

In other words, may not the quantities $\Sigma SPD(h_1, h_2 \dots h_n)$, &c., serve as natural and valuable intermediaries between the coefficients of an equation involving simple quantities and the coefficients of the equation involving the squares of their differences?

P.S. In the next part I trust to be able to present the readers of this *Magazine* with a *direct* and *symmetrical* method of eliminating any number of unknown quantities between any number of equations of any degree, by a newly invented process of symbolical multiplication, and the use of *compound* symbols of notation.

I must not omit to state that the constituents of multiplication λ_r and μ_r explained in Cor. 2 to Theorem 3 are equal to the expression

$$\Sigma(x-k_1)(x-k_2)\dots(x-k_{n-r-1})\frac{\begin{pmatrix}k_1, k_2 \dots k_{n-r-1}\\ -h_1, -h_2 \dots -h_m\end{pmatrix}}{\begin{pmatrix}k_1, k_2 \dots k_{n-r-1}\\ -k_{n-r} \dots -k_n\end{pmatrix}},$$

and its analogue respectively.