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ON DERIVATION OF COEXISTENCE. PART II. BEING THE THEORY OF SIMULTANEOUS SIMPLE HOMOGENEOUS EQUATIONS.

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Art. (1). We shall have constant occasion in this paper to denote different quantities by the same letter affected with different subscribed numerical indices.

Such a letter is to be termed a "Base."

Every character consisting of a base and an inferior index, this index is called an argument of the base, namely, the first, second, or nth argument, according as 1, 2, or in general n, be the number subscribed.

Art. (2). I use the symbol PD to denote the product of the differences of the quantities to which it is prefixed (each being to be subtracted *from* each that follows); thus

$PD\,(a, b, c)$ indicates $(b-a)(c-a)(c-b)$.

$PD\,(0, a, b, c)$ indicates $abc\,(b-a)(c-a)(c-b)$.

$PD\,(0, a, b, c \ldots l)$ indicates $abc \ldots l \times PD\,(a, b, c \ldots l)$.

Art. (3). For want of a better symbol I use the Greek letter ζ to denote that the product of factors to which it is prefixed is to be effected after a certain symbolical manner. This I shall distinguish as the zeta-ic product.

The symbol ζ will never be prefixed except to factors, each of which is made up of one or more terms, consisting solely of linear arguments of different bases, that is, characters bearing indices below but none above.

I am thereby enabled to give this short rule for zeta-ic multiplication: "Imagine all the inferior indices to become superior, so that each argument is transformed into a *power* of its base; multiply according to the rules of ordinary algebra; after the multiplication has been *done fully out* depress all the indices into their original position; the result is the zeta-ic product*."

* It is scarcely necessary to add that an analogous interpretation may be extended to any zeta-ic function whatever. Thus

$$\zeta\,(a_1+b_1)^2 = a_2 + 2a_1b_1 + b_2,$$

$$\zeta \cos\,(a_1) = 1 - \frac{a_2}{1\,.\,2} + \frac{a_4}{1\,.\,2\,.\,3\,.\,4}, \text{ \&c.}$$

Thus for example $\zeta(a_r, b_s)$ is the same as simply $a_r b_s$, but $\zeta(a_r, a_s)$ represents not $a_r a_s$ but a_{r+s}.

So in like manner

$$\zeta\{(a_h - b_k)(a_l - b_m)\}$$

$$= a_{h+l} - a_h b_m - b_k a_l + b_{m+k},$$

$$\zeta\{(a_1 - b_1)(a_1 - c_1)(b_1 - c_1)\}$$

$$= \text{the } \textit{depressed product} \text{ of } (a-b)(a-c)(b-c)$$

$$= \text{the } \textit{depressed} \text{ value of } a^2(b-c) + b^2(c-a) + c^2(a-b),$$

that is, $$= a_2 b_1 - a_2 c_1 + b_2 c_1 - b_2 a_1 + c_2 a_1 - c_2 b_1.$$

Art. (4). We shall have occasion in this part to combine the two symbols ζ, PD: thus we shall use

$$\zeta PD(a_1 b_1) \text{ to denote } \zeta(b_1 - a_1),$$

$$\zeta PD(a_1 b_1 c_1) \text{ to denote } \zeta\{(b_1 - a_1)(c_1 - a_1)(c_1 - b_1)\}.$$

Art. (5). For the sake of elegance of diction I shall in future sometimes omit to insert the inferior index when it is unity; but the reader must always bear in mind that it is to be *understood* though not expressed.

I shall thus be able to speak of the zeta-ic product of such and such bases mentioned by name.

Art. (6). We are not yet come to the limit of the powers of our notation. The zeta-ic product of the sum of arguments will consist of the sum of products of arguments, each argument being (as I have defined) made up of a base and an inferior index. Now we may imagine each index of every term of the zeta-ic product *after it is fully expanded* to be increased or diminished by unity, or each at the same time to be increased or diminished by 2, or each in general to be increased or diminished by r. I shall denote this alteration by affixing an r with the positive or negative sign to the ζ. Thus

$$\zeta(a_1 - b_1)(a_1 - c_1) \text{ being equal to } a_2 - a_1 c_1 + b_1 c_1 - b_1 a_1,$$

$$\zeta_{+1}(a_1 - b_1)(a_1 - c_1) \text{ is equal to } a_3 - a_2 c_2 + b_2 c_2 - b_2 a_2,$$

$$\zeta_{-1}(a_1 - b_1)(a_1 - c_1) \text{ is equal to } a_1 - a_0 c_0 + b_0 c_0 - b_0 a_0.$$

In like manner $\zeta PD(a, b, c)$ indicating

$$b_2 a_1 - b_2 c_1 + c_2 b_1 - c_2 a_1 + a_2 c_1 - a_2 b_1,$$

$\zeta_{\pm r} PD(a, b, c)$ indicates

$$b_{2\pm r} a_{1\pm r} - b_{2\pm r} c_{1\pm r} + c_{2\pm r} b_{1\pm r} - c_{2\pm r} a_{1\pm r} + a_{2\pm r} c_{1\pm r} - a_{2\pm r} b_{1\pm r}.$$

I shall in general denote $\zeta_{+r} PD(a, b, c \ldots l)$ *actually expanded* as the zeta-ic product of $a, b, c, \ldots l$ in its rth *phase*.

Art. (7). *General Properties of Zeta-ic Products of Differences.*

If there be made one interchange in the order of the bases to which ζ is prefixed, the zeta-ic product, in whatever phase it be taken, remains unaltered in magnitude, but changes its sign.

If in any *phase* of a zeta-ic product two of the bases be made to coincide, the expansion vanishes.

Let $\int_1$ be used, agreeably to the ordinary notation, to denote the sum of the quantities to which it is prefixed, $\int_2$ to denote the sum of the binary products, $\int_3$ of the ternary ones, and so on.

Thus let $\int_1 (a_1b_1c_1)$ or $\int_1 (a, b, c)$ indicate $a_1 + b_1 + c_1$,

and $\int_2 (a_1b_1c_1)$ or $\int_2 (a, b, c)$ indicate $a_1b_1 + a_1c_1 + b_1c_1$,

and $\int_3 (a_1b_1c_1)$ or $\int_3 (a, b, c)$ indicate $a_1b_1c_1$,

we shall be able now to state the following remarkable proposition connecting the several phases of certain the same zeta-ic products.

Art. (8). Let $a, b, c, \ldots l$, denote any number of independent bases, say $(n-1)$; but let the arguments of each base be periodic, and the number of terms in each period the same for every base, namely n, so that

$$a_r = a_{r+n} = a_{r-n}, \quad a_n = a_0 = a_{-n},$$

$$b_r = b_{r+n} = b_{r-n}, \quad b_n = b_0 = b_{-n},$$

$$c_r = c_{r+n} = c_{r-n}, \quad c_n = c_0 = c_{-n},$$

$$\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots$$

$$l_r = l_{r+n} = l_{r-n}, \quad l_n = l_0 = l_{-n},$$

r being any number whatever. Then

$$\zeta_{-1} PD (0, a, b, c \ldots l) = \zeta \{\textstyle\int_1 (a, b, c \ldots l) \; \zeta PD (0, a, b, c \ldots l)\},$$

$$\zeta_{-2} PD (0, a, b, c \ldots l) = \zeta \{\textstyle\int_2 (a, b, c \ldots l) \; \zeta PD (0, a, b, c \ldots l)\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\zeta_{-r} PD (0, a, b, c \ldots l) = \zeta \{\textstyle\int_r (a, b, c \ldots l) \; \zeta PD (0, a, b, c \ldots l)\}.$$

This proposition admits of a great generalization*, but we have now all that is requisite for enabling us to arrive at a proposition exhibiting under one *coup d'œil* every combination and every effect of every combination that can possibly be made with any number of coexisting equations of the first degree, containing any number of *repeated*, or to use the ordinary language of analysts, (variable or) unknown quantities.

* See the Postscript to this paper for *one* specimen.

For the sake of symmetry I make every equation homogeneous; so that to eliminate n repeated terms, no more than n equations will be required.

In like manner the problem of determining n quantities from n equations will be here represented by the case in which we have to determine the *ratios* of $(n+1)$ quantities from n equations.

Art. (9). *Statement of the Equations of Coexistence.*

Let there be any number of bases $(a, b, c \ldots l)$, and as many repeated terms $(x, y, z \ldots t)$, and let the number of equations be any whatever, say n. The system may be represented by the *type* equation

$$a_r x + b_r y + c_r z + \ldots + l_r t = 0,$$

in which r can take up all integer values from $-\infty$ to $+\infty$. The specific number of equations given will be represented by making the arguments of each base *periodic*, so that

$$a_r = a_{\mu n+r},\ b_r = b_{\mu n+r},\ c_r = c_{\mu n+r},\ \ldots\ l_r = l_{\mu n+r},$$

μ being any integer whatever.

Art. (10). *Combination of the given Equations.—Leading Theorem.*

Take $f, g, \ldots k$ as the *arbitrary* bases of new and absolutely independent but periodic arguments, having the same index of periodicity (n) as $a, b, c \ldots l$, and being in number $(n-1)$, that is, one fewer than there are units in that index.

The number of *differing* arbitrary constants thus *manufactured* is $n(n-1)$.

Let $Ax + By + Cz + \ldots + Lt = 0$ be the general *prime* derivative from the given equations, then we may make

$$A = \zeta PD(0, a, f, g \ldots k),$$
$$B = \zeta PD(0, b, f, g \ldots k),$$
$$C = \zeta PD(0, c, f, g \ldots k),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$L = \zeta PD(0, l, f, g \ldots k).$$

Art. (11). Cor. 1. *Inferences from the Leading Theorem.*

Let the number of equations, or, which is the same thing, the index of periodicity (n), be the same as the number of repeated terms $(x, y, z \ldots t)$, then one relation exists between the coefficients: this is found by making the $(n-1)$ new bases coincide with $(n-1)$ out of the old bases. We get accordingly, as the result of elimination,

$$\zeta PD(0, a, b, c \ldots l) = 0.$$

Art. (11). Cor. 2. Let the number of equations be one more than that of the given bases, there will then be two equations of condition. These are represented by preserving one new arbitrary base, as λ. The result of elimination being in this case

$$\zeta PD\,(0, a, b, c \dots l, \lambda) = 0.$$

Example. The result of eliminating between

$$a_1 x + b_1 y = 0,$$

$$a_2 x + b_2 y = 0,$$

$$a_3 x + b_3 y = 0,$$

is $\zeta PD\,(0, a, b, \lambda) = 0$, that is

$$\lambda_3 b_2 a_1 - \lambda_3 b_1 a_2 + \lambda_1 b_3 a_2 - \lambda_1 b_2 a_3 + \lambda_2 b_1 a_3 - \lambda_2 b_3 a_1 = 0,$$

from which we infer, seeing that λ_3, λ_2, λ_1 are independent,

$$b_2 a_1 - b_1 a_2 = 0,$$

$$b_3 a_2 - b_2 a_3 = 0,$$

$$b_1 a_3 - b_3 a_1 = 0,$$

any *two* of which imply the third.

In like manner, in general, if the number of equations exceed in any manner the number of bases or repeated terms, the rule is to introduce so many *new* and *arbitrary* bases as together with the old bases shall make up the number of equations, and then equate the zeta-ic product of the differences of zero, the old bases and the new bases, to nothing.

Art. (12). Cor. 3. Let the number of equations be *one* fewer than the number (n) of bases or repeated terms; the number of introduced bases in the general theorem is here ($n-2$). Make these ($n-2$) bases equal severally to the bases which in the type equation are affixed to $z, u \dots t$, then

$$C = 0,$$

$$D = 0,$$

$$\dots\dots$$

$$L = 0,$$

and we have left simply

$$\zeta PD\,(0, a, c, d \dots kl)\, x + \zeta PD\,(0, b, c, d \dots kl)\, y = 0.$$

In like manner we may make to vanish all but A and C, and thus get

$$\zeta PD\,(0, a, b, d \dots kl)\, x + \zeta PD\,(0, c, b, d \dots kl)\, z = 0,$$

and similarly

$$\zeta PD\,(0, a, b \ldots k)\,x + \zeta PD\,(0, b, c \ldots l)\,t = 0.$$

$$\text{Hence}\quad \left.\begin{matrix} x \\ y \\ z \\ . \\ . \\ . \\ t \end{matrix}\right\}\ \text{are severally as}\ \left\{\begin{matrix} \zeta PD\,(0, b, c \ldots l) \\ \zeta PD\,(a, 0, c \ldots l) \\ \zeta PD\,(a, b, 0 \ldots l) \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \zeta PD\,(a, b, c \ldots 0). \end{matrix}\right.$$

This is the symbolical representation as a *formula* of the remarkable *method* discovered by Cramer, perfected by Bezout and demonstrated by Laplace for the solution of simultaneous simple equations.

Art. (13). Cor. 4. In like manner if the number of repeated terms be two greater than the number of equations, we have for the relation between any *three* of them, taken at pleasure, for instance, x, y, z,

$$\zeta PD\,(0, a, d \ldots l)\,x + \zeta PD\,(0, b, d \ldots l)\,y + \zeta PD\,(0, c, d \ldots l)\,z = 0.$$

And in like manner we may proceed, however much in excess the number of repeated terms (unknown quantities) is over the number of equations.

Art. (14). *Subcorollary to Corollary* 3.

If there be any number of bases $(a, b, c \ldots l)$, and any other two fewer in number $(f, g \ldots k)$

$$\left.\begin{aligned} &\zeta PD\,(a, f, g \ldots k) \times \zeta PD\,(b, c \ldots l) \\ &+ \zeta PD\,(b, f, g \ldots k) \times \zeta PD\,(a, c \ldots l) \\ &+ \zeta PD\,(c, f, g \ldots k) \times \zeta PD\,(b, c \ldots l) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \zeta PD\,(l, f, g \ldots k) \times \zeta PD\,(a, b, c \ldots\ \ ) = 0 \end{aligned}\right\}^{*},$$

a formula that from its very nature suggests and *proves* a wide extension of itself.

In conclusion I feel myself bound to state that the principal substance of *Corollaries* (1), (2) and (3) may be found in Garnier's *Analyse Algébrique*, in the chapter headed "Dévéloppement de la Théorie donnée par M. Laplace, &c." But I am not aware of having been anticipated either in the fertile notation which serves to express them nor in the general theorems to which it has given birth.

P.S. I shall content myself for the present with barely enunciating a theorem, one of a class destined it seems to the author to play no secondary part in the development of some of the most curious and interesting points of analysis.

* The cross is used to denote *ordinary* algebraical multiplication.

Let there be $(n-1)$ bases $a, b, c \dots l$, and let the arguments of each be "recurrents of the nth order*," that is to say let

$$a_\iota = \phi\left(\cos\frac{2\pi\iota}{n}\right),\ b_\iota = \psi\left(\cos\frac{2\pi\iota}{n}\right),\ c_\iota = \chi\left(\cos\frac{2\pi\iota}{n}\right),\ \dots\dots l_\iota = \omega\left(\cos\frac{2\pi\iota}{n}\right).$$

Let R_r denote that any symmetrical function of the rth degree is to be taken of the quantities in a parenthesis which come after it, and let ϑ indicate any function whatever. Then the zeta-ic product

$$\zeta\{\zeta R_r(a, b, c \dots l) \times \zeta_\rho \vartheta PD(0, a, b, c \dots l)\}$$

is equal to the product of the *number*

$$R_r\left\{\left(\cos\frac{2\pi}{n} + \sqrt{(-1)}\sin\frac{2\pi}{n}\right), \left(\cos\frac{4\pi}{n} + \sqrt{(-1)}\sin\frac{4\pi}{n}\right), \left(\cos\frac{6\pi}{n} + \sqrt{(-1)}\sin\frac{6\pi}{n}\right),\right.$$

$$\left.\dots\dots, \left(\cos\frac{2(n-1)\pi}{n} + \sqrt{(-1)}\sin\frac{2(n-1)\pi}{n}\right)\right\},$$

multiplied by the zeta-ic phase

$$\zeta_{\rho-r}\vartheta PD(0, a, b, c \dots l)!!$$

* I am indebted for this term to Professor De Morgan, whose pupil I may boast to have been. I have the sanction also of his authority, and that of another profound analyst, my colleague Mr Graves, for the use of the arbitrary terms zeta-ic, zeta-ically. I take this opportunity of retracting the symbol *SPD* used in my last paper, the letter *S* having no meaning except for English readers. I substitute for it *QDP*, where *Q* represents the Latin word Quadratus. On some future occasion I shall enlarge upon a new method of notation, whereby the language of analysis may be rendered much more expressive, depending essentially upon the use of similar figures inserted within one another, and containing numbers or letters, according as quantities or operations are to be denoted. This system to be carried out would require special but very simple printing types to be founded for the purpose.

In the next part of this paper an easy and *symmetrical* mode will be given of representing any polynomial either in its developable or expanded form.