9.

A METHOD OF DETERMINING BY MERE INSPECTION THE DERIVATIVES FROM TWO EQUATIONS OF ANY DEGREE.

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LET there be two equations, one of the nth, the other of the mth degree in x; let the coefficients of the first equation be $a_n, a_{n-1}, a_{n-2} \dots a_0$, *each* power of x having a coefficient attached to it, a_n belonging to x^n and a_0 to the constant term.

In like manner let $b_m, b_{m-1} \dots b_0$ be the coefficients of the second equation.

I begin with

A Rule for absolutely eliminating x.

Form out of the (a) progression of coefficients m lines, and in like manner out of the (b) progression of coefficients form n lines in the following manner:

1. (a) Attach $(m-1)$ zeros all to the *right* of the terms in the (a) progression; next attach $(m-2)$ zeros to the right and carry over to the left; next attach $(m-3)$ zeros to the right and carry over 2 to the left. Proceed in like manner until all the $(m-1)$ zeros are carried over to the left and none remain on the right.

The m lines thus formed are to be written under one another.

1. (b) Proceed in like manner to form n lines out of the (b) progression by scattering $(n-1)$ zeros between the right and left.

2. If we write these n lines under the m lines last obtained, we shall have a solid square $(m+n)$ terms *deep* and $(m+n)$ terms *broad.*

3. Denote the lines of this square by arbitrary characters, which write down in vertical order and permute in every possible way, but separate the permutations that can be derived from one another by an even number of interchanges (effected between *contiguous* terms) from the rest; there will thus be half of one kind and half of another.

4. Now arrange the $(m+n)$ lines accordingly, so as to obtain

$$\tfrac{1}{2}\{(m+n)(m+n-1)\ldots 2\,.\,1\}$$

squares of one kind which shall be called positive squares, and an equal number of the opposite kind which shall be called negative.

Draw diagonals in the same direction in all the squares; multiply the coefficients that stand in any diagonal line together: take the sum of the diagonal products of the *positive* squares, and the sum of the diagonal products of the *negative* squares; the difference between these two sums is the prime derivative of the zero degree, that is, is the result of elimination between the two given equations reduced to its ultimate state of simplicity, there will be no irrelevant factors to reject, and no terms which mutually destroy.

Example. To *eliminate* between

$$ax^2+bx+c=0,$$

$$lx^2+mx+n=0,$$

I write down

$$a,\ b,\ c,\ 0, \qquad (1)$$

$$0,\ a,\ b,\ c, \qquad (2)$$

$$l,\ m,\ n,\ 0, \qquad (3)$$

$$0,\ l,\ m,\ n. \qquad (4)$$

I permute the four characters (1), (2), (3), (4), distinguishing them into positive and negative; thus I write together

Positive Permutations.

1	2	3	1	2	3	2	1	3	4	4	4
2	3	1	4	4	4	1	3	2	2	1	3
3	1	2	2	3	1	4	4	4	1	3	2
4	4	4	3	1	2	3	2	1	3	2	1

and again

Negative Permutations.

1	2	3	4	4	4	2	1	3	2	1	3
2	3	1	1	2	3	4	4	4	1	3	2
4	4	4	2	3	1	1	3	2	3	2	1
3	1	2	3	1	2	3	2	1	4	4	4

I reject from the permutations of each species all those where 1 or 3 appear in the fourth place, and also those where 2 or 4 appear in the first place, for these will be presently seen to give rise to diagonal products which are zero.

The permutations remaining are

Positive effectual permutations.

1	3	3	1
2	1	4	3
3	2	1	4
4	4	2	2

Negative effectual permutations.

3	1	1	3
1	4	3	2
4	3	2	1
2	2	4	4

I now accordingly form four positive squares, which are

$a, b, c, 0,$	$l, m, n, 0,$	$l, m, n, 0,$	$a, b, c, 0,$
$0, a, b, c,$	$a, b, c, 0,$	$0, l, m, n,$	$l, m, n, 0,$
$l, m, n, 0,$	$0, a, b, c,$	$a, b, c, 0,$	$0, l, m, n,$
$0, l, m, n,$	$0, l, m, n,$	$0, a, b, c,$	$0, a, b, c.$

Drawing diagonal lines from left to right, and taking the sum of the diagonal products, I obtain $a^2n^2 + lb^2n + l^2c^2 + am^2c$. Again, the four negative squares

$l, m, n, 0,$	$a, b, c, 0,$	$a, b, c, 0,$	$l, m, n, 0,$
$a, b, c, 0,$	$0, l, m, n,$	$l, m, n, 0,$	$0, a, b, c,$
$0, l, m, n,$	$l, m, n, 0,$	$0, a, b, c,$	$a, b, c, 0,$
$0, a, b, c,$	$0, a, b, c,$	$0, l, m, n,$	$0, l, m, n,$

give as the sum of the diagonal products

$$lbmc + alnc + ambn + lacn,$$

that is,
$$lbmc + ambn + 2acln.$$

Thus the result of eliminating between

$$ax^2 + bx + c = 0,$$
$$lx^2 + mx + n = 0,$$

ought to be, and is

$$a^2n^2 + l^2c^2 - 2acln + lb^2n + am^2c - lbmc - ambn = 0.$$

Rule for finding the prime derivative of the first degree, which is of the form $Ax - B$.

Begin as before, only attach one zero less to each progression; we shall thus obtain *not* a square, but an oblong broader than it is deep, containing $(m+n-2)$ rows, and $(m+n-1)$ terms in each row: in a word, $(m+n-2)$ rows, and $(m+n-1)$ *columns.*

To find A reject the column at the extreme right, we thus recover a square arrangement $(m+n-2)$ terms broad and deep.

Proceed with this new square as with the former one; the difference between the sums of the positive and negative diagonal products will give A.

To find B, do just the same thing, with the exception of striking off not the last column, but the last but *one.*

Rule for finding the prime derivative of any degree, say the rth, namely, $A_r x^r - A_{r-1} x^{r-1} + \ldots\ldots \pm A_0$.

Begin with adding zeros as before, but the number to be added to the (a) progression is $(m-r)$ and to the (b) progression $(n-r)$.

There will thus be formed an oblong containing $(m+n-2r)$ rows, and $(m+n-r)$ terms in each row, and therefore the same number of *columns.*

To find any coefficient as A_s, strike off all the last $(r+1)$ columns except that which is (s) places distant from the extreme right, and proceed with the resulting squares as before.

Through the well-known ingenuity and kindly proferred help of a distinguished friend, I trust to be able to get a machine made for working Sturm's theorem, and indeed all problems of derivation, after the method here expounded; on which subject I have a great deal more yet to say, than can be inferred from this or my preceding papers.