36.

AN ENUMERATION OF THE CONTACTS OF LINES AND SURFACES OF THE SECOND ORDER.

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It is well known that in general any two homogeneous quadratic functions of the same system of variables may be simultaneously transformed, so as to be expressed each of them as pure quadratic functions of a new system of variables equal in number and linearly connected with the original ones; a pure quadratic function meaning one in which only the squares of the variables are retained.

Every homogeneous quadratic function may be treated as the characteristic* of a locus of the second degree: if the function be of two letters, the locus is a binary system of points in a line wherein the distances of two fixed points from either point of the given system or given multiples of such distances correspond to the variables; if of three letters, the locus is a conic, the distances or given multiples of the distances of every point in which from three given lines in the plane of the conic are represented by the variables; if of four letters, the locus is a surface of the second order, the coordinates being the distances or multiples of the distances of any point therein from four planes drawn in the space in which the surface is contained, and so on for loci of four and higher dimensions.

I propose, however, in the present paper to restrict myself to the theory of the contacts of loci not transcending the limits of vulgar space, by which I mean the space cognizable through the senses†, and shall accordingly be

* According to the definition stated by me in a previous paper, the characteristic of a locus is the function which, equated to zero, constitutes the equation thereto.

† If the impressions of outward objects came only through the sight, and there were no sense of touch or resistance, would not space of three dimensions have been physically inconceivable? The geometry of three dimensions in ordinary parlance would then have been called transcendental. But in very truth the distinction is vain and futile. Geometry, to be properly understood, must be studied under a universal point of view; every (even the most elementary) proposition must be regarded as a fact, and but as a single specimen of an infinite series of homologous facts.

In this way only (discarding as but the transient outward form of a limited portion of an infinite system of ideas, all notion of extension as essential to the conception of geometry, however useful as a suggestive element) we may hope to see accomplished an organic and vital development of the science.

almost exclusively concerned in determining the singular cases of conjugate systems of quadratic forms of two, three, and four letters respectively.

In order that the reduction of any such system, say U and V, to a pure quadratic form may be possible (as it generally is), it is necessary that none of the roots of the complete determinant of $U + \lambda V$ shall be equal; if any relation of equality exist between these roots, the *general* reduction is *generally* no longer possible; under peculiar conditions, however, as will hereafter appear, in spite of the equality of certain of the roots, the irreducibility in its turn will cease, and the ordinary reduction be capable of being effected. It is easily seen, that to every relation of equality between the roots of the determinant of $U + \lambda V$ must correspond a particular species of contact between the loci which U and V characterize. But we should make a great mistake were we to suppose that every such relation of equality corresponded with but one species of contact; for instance, the characteristics of U and V of two conics are functions of three letters, and $\square(U + \lambda V)$ will be a cubic function of λ. Such a function may have two roots, or all its roots equal: this would seem to give but two species of contact, whereas we well know that there are no less than four species of contact possible between two conics. Accordingly we shall find, that, in order to determine the distinctive characters of each species of contact, we must look beyond the complete determinant, and examine into the relations (in themselves and to one another) of the several systems of minor determinants that can be formed from $U + \lambda V$.

By pursuing this method, we may assign *à priori* all the possible species of contact between any two loci of the second degree. How important this method is will be apparent from the fact, that not only have the distinctive characters of the various contacts possible between surfaces of the second order never been determined, but their number and the nature of certain of them have remained until this hour unknown and unsuspected.

The method which we shall pursue is an exhaustive one, and will conduct us by a natural order to a systematic arrangement of all the different modes and gradations of such contacts.

In a paper* in this *Magazine* for November 1850, I explained the decline of minor determinants, and stated a law, called the homaloidal law, concerning them.

If U and V be characteristics of the two loci whose contacts are to be considered, $U + \lambda V$ will be the function, the properties of whose complete determinant, and of the minor systems of determinants belonging to it, will serve to specify the nature of the contact.

It will be remembered, that, whatever be the number of variable letters in any quadratic function U, three of its first minor determinants being zero,

[* p. 150 above.]

makes all the first minors zero; six of its second minors being zero, makes all the second minors zero; and so on for the third, fourth, &c. minor systems according to the progression of the triangular numbers.

It is well known that whatever linear transformations be applied to a quadratic function W, the complete determinant thereof will remain unaltered, except by a multiplier depending upon the coefficients introduced into the equations of transformation; consequently the roots of λ in the equation obtained by making the determinant of $U+\lambda V$ zero remain unaffected by such transformation; and any relation or relations of equality among the roots of the equation $\Box(U+\lambda V)=0$ is an immutable property of the system U, V, which is unaffected by linear transformations. Another and more general kind of immutable property (comprehending the above as a particular case), to which I shall have occasion to refer, is the following.

Suppose all the minors of any order of $U+\lambda V$ have a factor $\lambda+\epsilon$ in common; this factor will continue common to the same system of minors when U and V are simultaneously transformed. This is a very important proposition, and easily demonstrated; for if $\lambda+\epsilon$ be a common factor to all the rth minors of $U+\lambda V$, $(U-\epsilon V)$ will have its rth minors zero, and therefore, as explained by me in the paper above referred to, $U-\epsilon V$ will be degraded r orders below U or V. This is clearly a property independent of linear transformation, consequently $\lambda+\epsilon$ will remain a factor of the transformed rth minors.

In like manner it is demonstrable that any number of *distinct* factors $\lambda+\epsilon_1$, $\lambda+\epsilon_2\ldots$ common to the rth minors of one form of $U+\lambda V$, will remain common factors of any other linearly derived form of the same. It is consequently necessary that each rth minor of one form of any quadratic function W shall be a syzygetic* function of *all* the rth minors of any other form of the same; and consequently a function of λ of any degree, whether all its factors be or *be not distinct*, which is common to the rth minors of one form of $U+\lambda V$, will remain so to the rth minors of any other form of the same.

The law exhibiting the connexion of each rth minor of one form of W (any homogeneous quadratic function) with all the rth minors of any other form of W, will form the subject of a distinct communication.

Finally, to fully comprehend the annexed discussion, the following principle must be apprehended.

* If $A=pL+qM+rN+\&c.$, where p, q, $r\ldots$ do not any of them become infinite when L, M, $N\ldots$ or any of them become zero, A may be termed a syzygetic function of L, M, $N\ldots$.

In the theorem above alluded to, it will be shown (as might be expected) that the syzygy in the case concerned is of the simplest kind, that is, that each rth minor of a quadratic function of any number of letters is a homogeneous linear function of all the rth minors of the same quadratic function linearly transformed.

If any factor K^e enter into all the rth minors of W, and if K^i be the highest power of K common to all the $(r+1)$th minors, then K^{2e-i} will be a common factor to all the $(r-1)$th minors.

Let r be taken unity; it is easily proved* that the complete determinant of *any* square matrix may be expressed by the difference between two products†, each of two first minor determinants divided by a certain second minor determinant. The proposition is therefore demonstrated for this case, and thereby in fact implicitly for every case, inasmuch as the first minors of every rth minor are $(r+1)$th minors of the original matrix. Hence it follows, that if any system of rth minor determinants have a common factor ϵ^i, the complete determinant must contain *at lowest* the factor $\epsilon^{(r+1)i}$, and any system of $(r-s)$th minor determinants thereunto will contain *at lowest* the factor $\epsilon^{(s+1)i}$.

I now proceed to apply these principles to the determination of the relative forms of conjugate quadratic functions representing geometrical loci of the second order. I shall begin with two binary systems of points in a right line.

The general characteristics U and V of two such systems may be thrown under the form

$$\left.\begin{aligned} U &= x^2 + y^2 \\ V &= ax^2 + by^2 \end{aligned}\right\}.$$

When $\square(V + \lambda U) = 0$ has its two roots equal, these systems have a point in common. The above forms cease to be applicable, and convert into

$$\left.\begin{aligned} U &= xy \\ V &= ax^2 + bxy \end{aligned}\right\}$$

where $x = 0$ represents the common point.

* This will appear in my promised paper on Determinants and Quadratic Functions.

† When the matrix is symmetrical about one of its diagonals (as it is in the case which we are concerned with), one of these products becomes a square. I may take this occasion of hinting, that the theory of quadratic functions merges in a larger theory of binary functions, consisting of the sum of the multiples of binary products formed by combining each of one set of quantities, x, y, z ... with each of the same number of quantities of another set, as x', y', z' .... For instance,

$$\begin{aligned} &axx' + bxy' + cxz' \\ &+ a'yx' + b'yy' + c'yz' \\ &+ a''zx' + b''zy' + c''zz' \end{aligned}$$

would be a binary function, and its determinant (no longer, as in a quadratic function, symmetrical about either diagonal) would correspond to the square matrix

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c''. \end{matrix}$$

Almost all the properties of quadratic apply, with slight modifications, to binary functions.

Let U and V now represent two conics. When there is no contact, we have as the types of their characteristics

$$U = x^2 + y^2 + z^2,$$
$$V = ax^2 + by^2 + cz^2.$$

The three roots of $\square (V + \lambda U) = 0$ are

$$\lambda = -a, \quad \lambda = -b, \quad \lambda = -c,$$

showing that there are three distinct pairs of lines in which the intersections of U and V are contained, the equations to three pairs being respectively

$$(b-a) y^2 + (c-a) z^2 = 0,$$
$$(c-b) z^2 + (a-b) x^2 = 0,$$
$$(a-c) x^2 + (b-c) y^2 = 0;$$

the four points of the intersection being defined by the equations corresponding to the proportions

$$x : y : z :: \sqrt{(b-c)} : \sqrt{(c-a)} : \sqrt{(a-b)}.$$

Now let $\square (U + \lambda V)$ have two equal roots; the characteristics assume the form

$$U = x^2 + y^2 + xz,$$
$$V = ax^2 + by^2 + cxz^*.$$

Two of the pairs of lines become identical, that is, two of the four points of intersection coincide.

* We may if we please make $a=b$; for it may be shown that the equations, in their present forms, contain an arbitrariness of 10 degrees; namely, 9 on account of x, y, z being arbitrary linears of ζ, η, θ; 2 on account of the ratios $a : b : c$; together 11 reduced by one degree on account of x, y, z, changed into lx, ly, lz, leaving $U=0$, $V=0$ unaffected. Now the degrees of arbitrariness in two conics, subject to satisfy only one condition, is $2 \times 5 - 1$ or 9. Hence there is one degree of arbitrariness to spare. In fact, on making $a=b$, the axis z becomes the line joining the two points of intersection distinct from the point of contact; x remaining the tangent at the point of contact, and y, strange to say, still arbitrary, subject only to passing through the point of contact; if, however, y be made to pass through the point of contact, and either one of the distinct intersections, this form,

$$U = x^2 + y^2 + xz,$$
$$V = ax^2 + ay^2 + cxz,$$

becomes no longer tenable, but gives place to

$$U = y^2 + yx + xz,$$
$$V = ay^2 + ayx + cxz,$$

where x is the tangent at the point of contact, z the line joining the two intersections with one another, and x, $x+y$ respectively the lines joining either of them with the point of contact; if the multiplier of yx in V in the above be made b instead of a, x remains the tangent as before, y becomes any line through the point of contact, and z any line through one of the distinct intersections. A systematic view of similar modulations of form and the study of the laws of arbitrariness connected with them, as applicable to the general subject-matter of this paper, must be deferred to a subsequent occasion.

This may be termed "Simple Contact." The tangent at the point of contact is $x=0$; this equation making U and V each become of only one order.

The intersections are

$$x=0, \quad y=0, \tag{1}$$

$$x=0, \quad y=0, \tag{2}$$

$$\sqrt{(a-c)}\,x+\sqrt{(b-c)}\,y=0, \quad z=0, \tag{3}$$

$$\sqrt{(a-c)}\,x-\sqrt{(b-c)}\,y=0, \quad z=0. \tag{4}$$

These are obtained by making $V-aU=0$, which gives $x=0$ or $z=0$.

$x=0$ gives $y^2=0$, that is, $y=0$ twice over, and $z=0$ gives

$$(a-c)\,x^2+(b-c)\,y^2=0.$$

The number of conditions to be satisfied in this case is one only.

Next let $\square(U+\lambda V)$ have all its roots equal. This condition will be satisfied (still leaving U and V as general as they can remain consistent with these conditions) by making

$$U=x^2+yz+yx,$$

$$V=ax^2+ayz+byx.$$

Here only one *distinct* pair of lines can be drawn to contain the intersections, showing that three out of the four points come together.

This may be termed "Proximal Contact." The number of affirmative conditions to be satisfied is two, and the contact is therefore entitled of the second degree.

The tangent at the point of contact is $y=0$, and the four intersections become

$$x=0, \quad y=0,$$

$$x=0, \quad y=0,$$

$$x=0, \quad y=0,$$

$$x=0, \quad z=0.$$

These may be obtained from the equation $V-aU=0$, which gives $y=0$ or $z=0$; the former implying concurrently with itself $x^2=0$, and the latter $yz=0$.

Thus we obtain three systems,

$$x=0, \quad y=0,$$

and one

$$x=0, \quad z=0,$$

corresponding to three consecutive points and the single distinct one.

The determinant of $U+\lambda V$ being only of the third degree in λ, we have exhausted the singularities of the system U, V dependent on the form of the complete determinant of $U+\lambda V$.

Let now the first minors of $U+\lambda V$ have a factor in common; this will indicate that $U+\lambda V$ may be made to lose *two* orders by rightly assigning λ, in other words, that the intersections of U and V are contained upon a pair of *coincident* lines. Here it is remarkable that the original forms of U and V reappear, but with a special relation of equality between the coefficients: we shall have, in fact,

$$U = x^2 + y^2 + z^2,$$
$$V = ax^2 + ay^2 + bz^2.$$

This gives the law for double, or, as I prefer to call it, diploidal contact*. By virtue of the Homaloidal law, we know that if three first minors of $U+\lambda V$ be zero, all are zero; we have therefore to express that three quadratic functions of λ have a root in common. This implies the existence of two *affirmative* conditions; the contact of the two conics taken collectively may therefore be still entitled of the second degree, although the contact at each of the two points where it takes place is simple, or of the first degree.

These points are evidently defined by the equation

$$\{x+\sqrt{(-1)}\,y=0,\ z=0\},$$
$$\{x-\sqrt{(-1)}\,y=0,\ z=0\},$$

and the ordinary algebraical solution of the equations $U=0$, $V=0$ would naturally lead to the four systems

$$x+\sqrt{(-1)}\,y=0, \quad z=0,$$
$$x+\sqrt{(-1)}\,y=0, \quad z=0,$$
$$x-\sqrt{(-1)}\,y=0, \quad z=0,$$
$$x-\sqrt{(-1)}\,y=0, \quad z=0;$$

the two tangents at the point of contact are $x+\sqrt{(-1)}\,y=0$, $x-\sqrt{(-1)}\,y=0$, and the coincident pair of lines containing the intersections is $z^2=0$.

* See my remarks† on the conditions which express double contact in the *Cambridge Journal*, Nov. 1850. If n functions, being all zero, be the condition of a fact, but r independent syzygetic equations admit of being formed between these functions, the number of affirmative conditions required is not n, but $(n-r)$; because the fact may be expressed by affirming $(n-r)$ equations and denying certain others. Thus if $P=0$, $Q=0$, $R=0$, $S=0$ express a fact, and

$$PP'+QQ'+RR'+SS'=0,$$
$$PP''+QQ''+RR''+SS''=0,$$

the fact is expressible by affirming $P=0$, $Q=0$, and denying $R'S''-R''S'=0$, for then $P=0$, $Q=0$ will imply $R=0$, $S=0$; or, in like manner, by affirming any other two out of the four necessary equations, and denying the other equations. Observe, however, that all the required equations *may* coexist in the absence of such right of denial.

[† p. 129 above.]

It may at first view appear strange, that whilst no condition is required in order that U and V may be simultaneously metamorphosed into the forms of $x^2+y^2+z^2$, $ax^2+by^2+cz^2$, a, b and c being all unequal, for this metamorphosis to be possible when any two become equal, not one but two conditions must be satisfied. The reason of this is, that the coefficients of transformation, which, as well as a, b, c, are functions of the coefficients of the given quadratic functions, become infinite on constituting between the said coefficients such relations as are necessary for satisfying the equation $a=b$, or $a=c$, or $b=c$, except upon the assumption of some further particular relations between them over and above that implied in such equality.

In the ordinary case of diploidal contact, the first minors having a factor in common, this factor will enter twice into the complete determinant of $U+\lambda V$, but it *may* enter three times: this will indicate, that not only do the four intersections lie on a coincident pair of lines, but furthermore, that there is but one pair of lines of any kind on which they lie.

In the ordinary case of diploidal contact, it will be observed that this latter condition does not obtain; the four intersections lie on a coincident pair of lines; but they lie also on a crossing pair, namely, in the two tangents at the points of contact. In this higher species of diploidal contact, it is clear that the two points of contact, which are ordinarily distinct, come together, and that all four intersections coincide.

This I call *confluent* contact; the forms of U and V corresponding thereto will be

$$U=x^2+y^2+xz,$$
$$V=ay^2+axz;$$

the common tangent at the point of contact being $x=0$, and the four coincident points $x^2=0$, $y^2=0$.

The number of affirmative conditions to be satisfied being three, the contact is to be entitled of the third degree.

Observe, that it is of no use to descend below the first minors in this case; because the second minors, being linear functions of λ, could not have a factor in common, unless $V:U$ becomes a numerical ratio, which would imply that the conics coincided*.

Fortified by the successful application of our general principles to the preceding more familiar cases of contact, we are now in a condition to apply with greater confidence the same *à priori* method to the exhaustion and characterization of all the varied species of contact possible between surfaces

* No-contact and complete coincidence may be conceived as the two extreme cases in the scale of relative conjugate forms.

of the second order; a portion of the subject comparatively unexplored, and never before thought susceptible of reduction to a systematic arrangement.

When there is no contact, we may write

$$U = x^2 + y^2 + z^2 + t^2,$$
$$V = ax^2 + by^2 + cz^2 + dt^2,$$

and the intersection of the surfaces will lie in each of the four cones,

$$(a-d)\,x^2 + (b-d)\,y^2 + (c-d)\,z^2 = 0,$$
$$(a-b)\,x^2 + (c-b)\,z^2 + (d-b)\,t^2 = 0,$$
$$(a-c)\,x^2 + (b-c)\,y^2 + (d-c)\,t^2 = 0,$$
$$(b-a)\,y^2 + (c-a)\,z^2 + (d-a)\,t^2 = 0.$$

Whenever the surfaces are in contact, certain of these cones will coincide with certain others, so that their number will be always less than four. Also, as we shall find in such event, they may degenerate into pairs of intersecting or coincident planes.

Let us begin with considering the cases of contact for which the first minors (and consequently *à fortiori* the minors inferior to the first) have no factor in common.

Here $\square\,(V + \lambda U)$ is a biquadratic function.

If λ have all its roots unequal, we have U and V as above given.

If two roots are equal, the characteristics assume the form

$$\left.\begin{aligned} U &= x^2 + y^2 + z^2 + xt \\ V &= ax^2 + by^2 + cz^2 + dxt \end{aligned}\right\}.$$

The touching plane is $x = 0$; the point of contact is $x = 0$, $y = 0$, $z = 0$; the curve of intersection is one of the fourth degree, with a double point at the point of contact.

There is but one condition to be satisfied, and the contact may be entitled "simple" and of the first degree.

Next let λ have three equal values, the equations become

$$U = x^2 + yz + t^2 + xy,$$
$$V = x^2 + yz + at^2 + bxy.$$

The tangent plane at the point of contact $y = 0$, and the point itself $x = 0$, $y = 0$, $t = 0$. The curve of intersection is a curve of the fourth order, with a cusp at the point of contact. The number of affirmative conditions to be satisfied is two; the contact is of the second degree, and may be termed "proximal" or cuspidal.

Next let $\square(U + \lambda V)$ have two pairs of equal roots, we shall find

$$U = x^2 + xy + zt,$$

$$V = ayz + bxy + czt.$$

The line $x = 0$, $z = 0$ will be common to both surfaces. The curve of intersection will therefore break up into a right line and a line of the third order.

The former will meet the latter in two points, which will be each of them points of contact. The contact is therefore diploidal; but as there is another species of diploidal contact to which we shall presently come, it will be expedient to characterize each of them by the nature of the intersections of the two surfaces; accordingly this may be termed unilinear-intersection contact, or more briefly, unilinear contact.

The number of affirmative conditions to be satisfied being two, it may be said to be collectively of the second degree, but (obviously ?) the contact at each of the two points is of the nature of simple contact.

Lastly, let us suppose that all four roots of $U + \lambda V$ are equal; we shall find, as the most simple expressions of the most general forms of the two surfaces,

$$U = x^2 + xy + yz + zt,$$

$$V = axy + bz^2 + azt.$$

In this case the two points of intersection of the curve of the third degree, and the right line on which the surfaces intersect, come together, so that the right line becomes a tangent to the curve. The number of conditions to be satisfied is three: there is but one point of contact which may be considered as the union of two which have coalesced, and the species may be defined as confluent-unilinear contact.

If we throw the equations to the conoids having an unilinear contact into the form

$$x(x + y) + zt = 0,$$

$$xy + z(y + ct) = 0,$$

we obtain

$$(x + y)(y + ct) - yt = 0,$$

which last equation is no longer satisfied by $x = 0$, $z = 0$, these systems of roots having been made to disappear by the process of elimination.

The curve of the third degree, in which the two given conoids intersect, may thus be defined as their common intersection with the new conical surface defined by the third of the above equations.

More generally, it is apparent that the three conoids,

$$\left.\begin{array}{r} xu - yt = 0 \\ yv - zu = 0 \\ zt - xv = 0 \end{array}\right\},$$

in which x, y, z, t, u, v may any of them be considered as a homogeneous linear function of four others, intersect in the same line of the third degree. *Besides* which, the first and second intersect in the right line y, u; the second and third in z, v; the third and first in x, t; each of which lines it is evident is a chord of the common curve of intersection. For instance, $y = 0$, $u = 0$ may be satisfied concurrently with all the above three equations by satisfying the equation $zt - xv = 0$, which, as two linear relations exist originally between the six letters, and two more have been thrown in, becomes a quadratic equation between any two of the letters.

The only case of exception to this reasoning is, when $y = 0$, $u = 0$ can be satisfied concurrently with $z = 0$, $v = 0$, and with $x = 0$, $t = 0$; but in this case the surfaces all become cones; and as there is no longer a curve of the third degree, "Cadit quæstio." Even here, however, the intersection of any two of the surfaces becomes a conic, and two coincident generating lines on the two cones; so that if we take one of these and the conic to represent a degenerate form of a line of the third degree, the remaining straight line passes through a double point of such degenerate form, and the case passes into that of confluent-unilinear contact.

The two double points in the intersection of the two conoids

$$U = x(x + y) + zt = 0,$$

$$V = xy + z(y + ct) = 0,$$

by which I mean the points of intersection of the conic with the right line common to them, are found by making $x = 0$, $z = 0$, and substituting in the derived equation

$$(x + y)(y + ct) - ty = 0,$$

which gives $y = 0$, or $y + (c - 1)t = 0$; so that the two points required are

$$x = 0, \quad y = 0, \qquad z = 0,$$

$$x = 0, \quad y = (1 - c)t, \quad z = 0.$$

It appears also that the entire intersection is contained in each of the two cones,

$$U - V, \text{ that is, } x^2 + z\{(1 - c)t - y\}$$

and

$$cU - V, \text{ that is, } cx^2 + y\{(c - 1)x - z\},$$

the respective vertices of which are at the points above determined.

The equations for confluent-unilinear contact,

$$x(x+y)+z(y+t)=0,$$
$$xy+z(cz+t)=0,$$

give

$$(x+y)(cz+t)-(y+t)y=0;$$

which, on making $x=0$, $z=0$, is satisfied by $y^2=0$; showing that the *confluence* takes place at the point

$$x=0, \quad y=0, \quad z=0.$$

The number of terms in the two equations for ordinary unilinear contact being six, and in those given for confluent unilinears seven, and the empirical rule in all other cases being that the terms tend to diminish and never increase in number as the degree of the contact (expressed by the number of conditions to be satisfied) rises, I am led to suspect that the conjugate system for the latter species of contact may admit of being reduced to some more simple form.

I must state here once for all, that all the *distinct* systems of (at least consecutive) conjugate forms that have been, and will be given, are *mutually* untransformable. This it is which distinguishes *singular* from *particular* forms.

A particular form is included in its primitive; but a singular form is one, which, while it responds to the same conditions as some other *more general* form, is incapable of being expressed as a particular case of the latter, on account of the *additional* condition or conditions which attach to it.

I pass now to the singularities which arise from the first minor determinants of $U+\lambda V$ having a factor in common, the second minors being supposed to be still without a common factor.

When this common factor is linear in respect to λ, let it be supposed to enter not more than twice (twice, we know, by the general principle enunciated at the commencement of this paper, it must enter) into the complete determinant.

Two of the cones containing the intersection of U and V then become coincident, and degenerate each into the same pair of crossing planes. This may be termed biplanar-contact. The characteristics of such contact are

$$U=x^2+y^2+z^2+t^2,$$
$$V=ax^2+ay^2+bz^2+ct^2;$$

the points of contact are two in number, being at the intersection of the two plane conics into which the curve of intersection breaks up. The two planes in which these lie are given by the equation $(b-a)z^2+(c-a)t^2=0$; these intersect in the right line $z=0$, $t=0$, which meets both surfaces in the same two points,

$$z=0, \quad t=0, \quad x+\sqrt{(-1)}\,y=0,$$
$$z=0, \quad t=0, \quad x-\sqrt{(-1)}\,y=0,$$

the two common tangent planes at these points being

$$x+\sqrt{(-1)}\,y=0, \quad x-\sqrt{(-1)}\,y=0$$

respectively.

This, then, is another species of double contact between two conoids, and, as far as I know, the only kind hitherto recognized as such. The number of conditions to be satisfied remains two, as in the former species.

Next suppose that the common factor of the first minor enters three times into the complete determinant instead of *twice* only, as in the last case.

The corresponding characteristics will be found to be

$$U=x^2+zt+y^2+z^2,$$
$$V=ax^2+azt+by^2+cz^2.$$

The intersection of U, V still lies in two planes,

$$(b-a)y^2+(c-a)z^2=0;$$

but the intersection of these two planes,

$$y=0, \quad z=0,$$

meets the surfaces in the two coincident points,

$$y=0, \quad z=0, \quad x^2=0.$$

This, therefore, I call confluent-biplanar contact; the two conics constituting the complete intersection, instead of cutting, touch and at their point of contact the two conoids have a contact of a superior order. The conditions to be satisfied for this case are three in number.

Next suppose that the common factor of the first minors enters only twice into the complete determinant, but that the remaining two factors become equal.

Here the analytical characters of unilinear and biplanar contact are blended; in fact, the intersection consists of a conic and a pair of right lines meeting one another and the conic. The characteristics are

$$U=x^2+y^2+z^2+zt,$$
$$V=ax^2+ay^2+bz^2+czt.$$

The intersection is contained in the two planes

$$z=0, \quad (b-a)z+(c-a)t=0,$$

and consists of the two lines $z=0$, $x^2+y^2=0$, lying in the common tangent plane $z=0$, and the conic

$$\left.\begin{array}{l}(b-a)z+(c-a)t=0\\ (a-c)x^2+(a-c)y^2+(b-c)z^2=0\end{array}\right\}.$$

There are *three* points of contact, namely, the point $x=0$, $y=0$, $z=0$, where the two right lines cut, and $x^2+y^2=0$, $t=0$, $z=0$, where these lines meet the conic. This, then, is a case of triple contact. I distinguish it by the name of bilinear-contact. The number of conditions is still three.

Now all else remaining as before, let the two pairs of equal roots in the complete determinant become identical, or, in other words, let the common factor of the first minors be contained four times in the complete determinant. The characteristics become

$$U=xz+xt+y^2+z^2,$$
$$V=axz+bxt+by^2+bz^2.$$

The intersection becomes the two right lines

$$x=0, \quad y^2+z^2=0,$$

and the conic

$$z=0, \quad x^2+y^2=0.$$

All these meet in the same point,

$$x=0, \quad y=0, \quad z=0;$$

so that instead of contact in three points, the contact takes place about one only, in which the *three* may be conceived as merging. This I call confluent-bilinear contact. It requires the satisfaction of four conditions.

Next let us suppose that the two distinct factors are common to each of the first minors. This will imply the existence of four affirmative conditions.

The complete determinant will of necessity contain each of these factors twice, so that no additional singularity can enter through this determinant. The characteristics assume the form

$$U=x^2+y^2+z^2+t^2,$$
$$V=ax^2+ay^2+bz^2+bt^2.$$

The two surfaces will meet in four straight lines, forming a wry quadrilateral, whose equations are

$$x\pm\sqrt{(-1)}\,y=0,$$
$$z\pm\sqrt{(-1)}\,t=0.$$

These intersect each other in the four points

$$x = 0, \quad y = 0, \quad z^2 + t^2 = 0,$$
$$z = 0, \quad t = 0, \quad x^2 + y^2 = 0,$$

each of which will be a distinct point. This I term quadrilinear contact.

Now let the two factors common to each of the first minors become identical; so that a *squared* function, instead of an ordinary *quadratic* function of λ, is now their common measure.

The factor which enters twice into each of the first minors will enter four times into the complete determinant; the number of conditions to be satisfied is one more than in the preceding case, namely five, and the characteristics become

$$U = x^2 + y^2 + xz + yt,$$
$$V = ax^2 + by^2 + cxz + cyt.$$

Here arises a singularity of form in the intersections utterly unlike anything which has been remarked in the preceding cases. For it will not fail to have been observed, that the intersection in the nine preceding cases was always a line or system of lines of the fourth degree, so as to be cut by any plane in four points.

But in this case, the fact of the first minors having a factor in common, shows that the intersection is contained in two planes (which is of course to be viewed as a degenerate species of cone); and the fact of the complete determinant having all its roots equal, shows that there is but one system of a pair of planes in which the intersection is contained, and no more.

So that the two pairs of planes, into which the wry quadrilateral was divisible in the case immediately preceding, now become a single pair. This can only be explained by two of the opposite sides of the quadrilateral becoming indefinitely near to one another, but still not coinciding in the same planes; so that the actual *visible* or quasi-visible* intersection will be in three right lines, of which the middle one meets each of the two others.

This will further appear by proceeding regularly to solve the equations

$$U = 0, \quad V = 0.$$

$V - cU = 0$ gives $y = \pm kx$, where $k = \sqrt{\left(\dfrac{a-c}{c-b}\right)}$, and therefore $xz + kxt = 0$, or $xz - kxt = 0$; whence we see that the complete intersection is represented by the lines

$$(x = 0, \ y = 0); \quad (z + kt = 0, \ y - kx = 0),$$
$$(x = 0, \ y = 0); \quad (z - kt = 0, \ y + kx = 0),$$

* I use the term quasi-visible, because the intersection may become in part or whole imaginary.

showing that there are but three physically distinct lines, as already premised.

This, then, may be considered as derived from the preceding case of a wry quadrilateral intersection, by conceiving two opposite sides of the quadrilateral to come indefinitely near, but without coinciding.

Let these two lines be called P and P'; take any point in P and any two points in P' indefinitely near to one another and the point first taken, then this indefinitely small plane will be common to both surfaces, and consequently they ought to touch along *every point in the line P*. This is again confirmed by the forms given to U and V. For at any point where the coordinates are $0, 0, \zeta, \theta$ the equations to the tangent planes to the two surfaces respectively are

$$\zeta x + \theta y = 0,$$
$$c\zeta x + c\theta y = 0,$$

that is to say, are identical.

Whilst, therefore, certain grounds of geometrical, and still stronger grounds of analytical analogy, might seem to justify this species of contact taking the name of confluent quadrilinear, yet as, in fact, the intersection is trilinear, and as, moreover, the two indefinitely proximate lines must be considered, not as coincident, but as turned away from one another through an indefinitely small angle and out of the same plane, I prefer to take advantage of this striking property of contact *at every point* along a line (a property entirely distinct from any that we have yet considered), and confer upon the species of contact we have been considering the designation of unilinear-indefinite contact.

Where the line of indefinite contact meets the two other lines of the intersection, the contact is of course of a higher order; thus offering a parallel to what takes place in ordinary unilinear contact, in which there is no contact, except only *at two points* of the right line forming part of the complete intersection.

I believe that this kind of contact, which forms a natural family with two others about to be described, and which will close the list, has never before been imagined, and would at first sight have been rejected as impossible.

Having now exhausted the cases of the first class, in which the minors have no factor in common, and the two sections of the second class, in which the second minors have no common factor, but the first minors of $U + \lambda V$ a linear or quadratic function of λ in common, I descend to the third class, in which the second minors, which are quadratic functions of λ, are supposed to have a common factor.

This common factor must enter twice into each of the first minors by virtue of the law previously indicated, and cannot enter more than twice, as otherwise the first minors of $U+\lambda V$ could only differ from one another by a numerical multiplier, which is obviously impossible, except when $U+\lambda V$ is of the form $(k+\lambda)U$, that is, when the two surfaces coincide.

Again, the common factor of the first minor must enter three times into the complete determinant; but there is no reason why it may not enter four times, and thus two cases arise. In the first, the characteristics take the form

$$U = x^2 + y^2 + z^2 + t^2,$$
$$V = ax^2 + ay^2 + az^2 + bt^2.$$

The second determinant having a factor in common, shows that the intersection U, V is contained in a pair of coincident planes; but the complete determinant, having two distinct factors, evidences that these plane intersections, viewed as indefinitely near but still distinct, lie in the same cone, which will be a cone enveloping both the surfaces U and V all along their mutual intersections. This is also seen easily from the forms of U and V; for we have $V - aU = (b-a)t^2$, which proves that the intersection lies in the coincident, or, to speak more strictly, consecutive planes $t^2 = 0$; and at any point $x = \xi$, $y = \eta$, $z = \zeta$, the tangent plane to each surface becomes

$$\xi x + \eta y + \zeta z = 0.$$

As there are six independent, that is, non-necessarily co-evanescent second minors, that the second minor systems shall all have a common factor, implies the satisfaction of five conditions. This species of contact I call curvilineo-indefinite; it is, I believe, the only kind of indefinite contact between two surfaces of the second order hitherto taken account of.

There is still, however, a higher species of contact, *videlicet*, when all the four roots of the complete determinant of $U+\lambda V$ are identical with the root common to each of its second minors. In this case the common enveloping cone becomes identical with the plane (considered as a coincident pair of planes) in which the surfaces intersect.

The characteristics take the form

$$U = x^2 + xy + zt,$$
$$V = \qquad xy + zt.$$

The intersection is contained completely in the common tangent plane $x = 0$, and consists of the *two* right lines,

$$(x = 0, \quad z = 0),$$
$$(x = 0, \quad t = 0).$$

This, the highest and crowning species of contact, I call bilineo-indefinite. It is defined by six conditions.

At each point of the two lines of intersection of U and V there is contact, and a very peculiar species of contact at the intersection of these two lines themselves.

To form a distinct idea of this, let the physical visible or quasi-visible intersection of U, V take place along the two lines L, M; the *rational* intersection must be conceived as made up of the wry quadrilateral, L, M; L', M', in which L is indefinitely near to L', and M to M'. It follows, therefore, that there is contact at the four angles of the quadrilateral; but as there is nothing to fix the relative directions of the diagonal joining the intersection of L and M to that of L' and M', because there is nothing to restrict the position of the latter point, except that it shall lie upon either surface*, it appears that not only is there contact at the junction of the two lines constituting the complete intersection of the two surfaces, but that these surfaces continue to touch at consecutive points taken all round this first, and indefinitely near to it in any direction†.

Bilineo-indefinite (the highest) contact for two conoids is strictly analogous to confluence, the highest species of contact between conics. For this latter may be conceived as an intersection made up of two coincident pairs of coincident points; and the former, as an intersection made up of two coincident pairs of crossing right lines; and a pair of crossing lines is to a plane locus of the second degree what a coincident pair of points is to a rectilinear locus of the same degree.

In the subjoined table I have brought under one point of view the characters and algebraic forms which I call the *condensed forms* corresponding to each species of contact above detailed.

A. *Quadratic loci in a right line.*

Simple contact. One condition.	xy $x^2 + xy$

B. *Quadratic loci in a plane.*

1st Class.

Simple contact. One condition.	$x^2 + y^2 + xz$ $ax^2 + by^2 + cxz$
Proximal contact. Two conditions.	$x^2 + yx + yz$ $ax^2 + byx + ayz$

2nd Class.

Diploidal contact. Two conditions.	$x^2 + y^2 + z^2$ $ax^2 + ay^2 + bz^2$
Confluent contact. Three conditions.	$x^2 + y^2 + xz$ $y^2 + xz$

* This will be better seen by reference to the analogy presented by the case when the two conoids touch all along a curve. The rational intersection is made up of this curve and another indefinitely near it. The two curves, whatever be the position of their node, will lie in the same enveloping cone, so that the position of the node is indeterminate.

† As the two surfaces jut one close into the other at this point, it would perhaps be not improper to designate the contact at such point as umbilical.

C. *Quadratic loci in space.*

1st Class.

Simple contact. One condition.	$\left.\begin{matrix} x^2+y^2+z^2+xt \\ ax^2+by^2+cz^2+dxt \end{matrix}\right\}$
Proximal contact. Two conditions.	$\left.\begin{matrix} x^2+y^2+xt+zt \\ ax^2+by^2+cxt+azt \end{matrix}\right\}$
Unilinear contact. 1st species of diploidal. Two conditions.	$\left.\begin{matrix} x^2+xy+zt \\ ayz+bxy+czt \end{matrix}\right\}$
Confluent-unilinear, or triple contact. Three conditions.	$\left.\begin{matrix} x^2+yz+xy+zt \\ az^2+bxy+bzt \end{matrix}\right\}$

2nd Class, 1st Section.

Biplanar contact. 2nd species of diploidal. Two conditions.	$\left.\begin{matrix} x^2+y^2+z^2+t^2 \\ ax^2+ay^2+bz^2+ct^2 \end{matrix}\right\}$
Confluent-biplanar contact. Three conditions.	$\left.\begin{matrix} x^2+zt+y^2+z^2 \\ ax^2+azt+by^2+cz^2 \end{matrix}\right\}$
Bilinear contact. Three conditions.	$\left.\begin{matrix} x^2+y^2+z^2+zt \\ ax^2+ay^2+bz^2+czt \end{matrix}\right\}$ or $\left\{\begin{matrix} xz+yt \\ axt+byz \end{matrix}\right.$
Confluent-bilinear contact. Four conditions.	$\left.\begin{matrix} xz+xt+y^2+z^2 \\ axz+bxt+by^2+bz^2 \end{matrix}\right\}$

2nd Class, 2nd Section.

Quadrilinear, or quadruple contact. Four conditions.	$\left.\begin{matrix} x^2+y^2+z^2+t^2 \\ ax^2+ay^2+bz^2+bt^2 \end{matrix}\right\}$ or $\left\{\begin{matrix} xy+zt \\ axy+bzt \end{matrix}\right.$
Unilineo-indefinite contact. Five conditions.	$\left.\begin{matrix} x^2+y^2+xz+yt \\ ax^2+by^2+cxz+cyt \end{matrix}\right\}$

3rd Class.

Curvilineo-indefinite contact. Five conditions.	$\left.\begin{matrix} x^2+y^2+z^2+t^2 \\ ax^2+ay^2+az^2+bt^2 \end{matrix}\right\}$
Bilineo-indefinite contact. Six conditions.	$\left.\begin{matrix} x^2+xy+zt \\ xy+zt \end{matrix}\right\}$

Another (and, in a physical sense, more) natural mode of grouping the twelve species of conoidal contact, which, without observing the same lines of demarcation, leaves intact the sequence of the species, is into the three families. The first, or definite-continuous, for which the surfaces touch in a single point, and intersect in an unbroken curve, comprises simple and cuspidal contact.

The second definite-discontinuous, for which the surfaces touch in one, two, three or four points, but intersect in a curve more or less broken up into distinct parts, comprises all the species from the third to the ninth inclusive. The third natural family is that of indefinite contact, and comprises the three last species. It will of course be observed that there are five species of single contact, that is, contact at one point, namely, simple, cuspidal, and the three confluent species, two of double, one of treble, one of quadruple, and three of indefinite contact; the last being distinguishable *inter se*—lineo-indefinite as being special at two points, curvilineo-indefinite as having no speciality, and bilineo-indefinite as being special at one point only.

I might now proceed to discuss more particularly the nature of the contact taken, not collectively, but with reference to each single point where it exists. This, however, must be reserved for a future communication; as also, among other important and curious matter, the ascertainment of the singular forms of quadratic conjugate functions of five or more letters. At present I shall content myself with stating the following general proposition, which naturally suggests itself from a consideration of the cases already considered.

In a conjugate quadratic system of any number of letters, the lowest and also the highest degree of singularity will be always unique; the conditions to be satisfied in the former case being only one in number, and in the latter $\frac{1}{2}r(r-1)$, where r denotes the number of the letters. The first part of this proposition is self-apparent, the latter part may be inferred from the homaloidal law; for the $(r-2)$nd minors will be quadratic functions, and the highest degree of contact will correspond to those having a factor in common, which would involve the satisfaction of $\frac{1}{2}r(r-1)-1$ conditions only; but over and above this, that the complete determinant, instead of containing this common factor, as it needs must, $(r-1)$ times, shall contain it r times: this gives one condition more, making up the entire number to $\frac{1}{2}r(r-1)$.

The total number of different species of singularity for conjugate functions of a given number of letters, can only be expressed by aid of formulæ containing expressions for the number of various ways in which numbers admit of being broken up into a given number of parts.

The computation of this number in particular cases, upon the principle of the foregoing method, is attended with no difficulty.

We have seen that this number for two, three and four letters, is respectively one, four, twelve.

I have found that for five letters the number is twenty-four, for six letters fifty, for seven letters a hundred, and (subject to further examination) for eight letters one hundred and ninety-three. The series, therefore, as far as I have yet traced it, is 1, 4, 12, 24, 50, 100, 193. The last number must not be relied upon at present.

It will be observed, that the foregoing table for the contacts of surfaces of the second order contains no form corresponding to a complete intersection in two non-intersecting lines and an undegenerated conic. In fact, if two such lines form part of the intersection, *at least* one other right line intersecting them both, must go to make up the remaining part. This is easily verified; for it is readily seen that the most general representation of two conoids intersecting in two non-meeting lines will be

$$U = xy + zt,$$

$$V = axy + bzt + cxt + eyz,$$

where the two lines in question are

$$(x = 0, \quad z = 0),$$

$$(y = 0, \quad t = 0).$$

Now it will be found that the first minors of $V + \lambda U$ formed from the above equation will all contain the common factor $(a + \lambda)(b + \lambda) - ce$, showing that the contact is quadrilinear or linear-indefinite, that is bilinear, according as the roots of

$$\lambda^2 + (a + b)\lambda + ab - ce = 0$$

are distinct or equal; which explains how it is that only one species of bilinear contact (that is to say, the case corresponding to the two conoids agreeing in the two right lines in which each is cut by a common tangent plane) comes to find a place in the preceding enumeration.

It may not be uninteresting, under an euristic point of view, to state that the above theory, which, as well in what it accomplishes as in what it suggests (the author cannot but feel conscious), constitutes a substantial accession to analytical science, arose out of a theorem which occurred to him as likely to be true, in the act of reviewing for the press his paper "On Certain Additions" in the last November Number* of this *Magazine*, and which he had only then time to throw into a foot-note as a probable conjecture.

Wishing to subject it to an analytical test, he found it necessary to obtain the *condensed forms* which serve to characterize the confluent contact of

[* p. 148 above.]

conics. In this way he became aware of the great utility of these condensed forms, and of the desideratum to be supplied in obtaining a complete list of them applicable to all varieties of contact. The happy thought then occurred to him of inverting the process which he had applied in the treatment of the contacts of conics, in the November Number* of the *Cambridge and Dublin Mathematical Journal*; for whereas the nature of the contacts was there assumed and translated into the language of determinants, he soon discovered that it was the more easy and secure course to assume the relations of every possible *immutable* kind that could exist between the complete and minor determinants corresponding to the characteristics, by aid of these relations to construct the characteristics, and from the characteristics so obtained, determine the geometrical character of each resulting species of contact. Thus he has been able to effect the very results stated by himself as desiderata at the close of the paper in this *Magazine* above referred to.

Note.—It is proper to remark, that all the condensed forms given in this paper have actually been obtained by the author in the way above pointed out. The limits imposed by the objects to which the *Magazine* is devoted have restricted him from exhibiting the method at full; but any of his readers will be able without difficulty to make it out for himself.

The process consists in finding $U+\lambda V$ by means of solving for each case a problem of position (a kind of chess-board problem) on a square table, containing three places in length and breadth for conics, four places by four for surfaces, and so on (if need be) according to the number of variable letters involved. $U+\lambda V$ being thus determined in form, U and V become readily cognizable. It is right also to add, that some of the condensed forms here set forth have been incidentally noticed and employed by previous authors, as Plücker and Mr Cayley.

The conditions in each case to which the position-problem is subject are immediately deducible from the laws which the complete determinant, and the successive minor systems of determinants of $U+\lambda V$, are required to satisfy.

[* p. 119 above.]