40.

ON EXTENSIONS OF THE DIALYTIC METHOD OF ELIMINATION.

[*Philosophical Magazine*, II. (1851), pp. 221—230.]

THE theory about to be described is a natural extension of the method of elimination presented by me ten years ago (in June, 1841) in the pages of this *Magazine*, which I have been induced to review in consequence of the flattering interest recently expressed in the subject by my friend M. Terquem, and some other continental mathematicians, and because of the importance of the geometrical and other applications of which it admits, and of the inquiries to which it indirectly gives rise. We shall be concerned in the following discussion with systems of homogeneous rational integral functions of a peculiar form, to which for present purposes I propose to give the name of aggregative functions, consisting of ordinary homogeneous functions of the same variables but of different degrees, brought together into one sum made homogeneous by means of powers of new variables entering factorially.

Thus if $F, G, H \ldots L$ be any number of functions of any number of letters $x, y \ldots t$ of the degrees $m, m-\iota, m-\iota' \ldots m-(\iota)$ respectively,

$$F + G\lambda^{\iota} + H\mu^{\iota'} + \ldots + L\theta^{(\iota)}$$

will be an aggregative function of the variables entering into F, G, &c. and of $\lambda, \mu \ldots \theta$. I shall further call such a function binary, ternary, quaternary, and so forth, according to the number of variables contained in the functions F, G, H, &c. thus brought into coalition.

It will be convenient to recall the attention of the reader to the meaning of some of the terms employed by me in the paper above referred to.

If F be any homogeneous function of $x, y, z \ldots t$, the term augmentative of F denotes any function obtained from F of the form

$$x^{\alpha} y^{\beta} z^{\gamma} \ldots t^{\delta} \times F.$$

Again, if we have any number of such functions $F, G, H \ldots K$ of as many variables $x, y, z \dots t$, and we decompose $F, G, H \dots K$ in any manner so as to obtain the equations

$$F = x^a P_1 + y^b P_2 + z^c P_3 + \&c. \dots + t^d (P),$$

$$G = x^a Q_1 + y^b Q_2 + z^c Q_3 + \&c. \dots + t^d (Q),$$

$$H = x^a R_1 + y^b R_2 + z^c R_3 + \&c. \dots + t^d (R),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$K = x^a S_1 + y^b S_2 + z^c S_3 + \&c. \dots + t^d (S),$$

and then form the determinant

$$\begin{vmatrix} P_1, & P_2, & P_3 \dots (P) \\ Q_1, & Q_2, & Q_3 \dots (Q) \\ R_1, & R_2, & R_3 \dots (R) \\ \dots & \dots & \dots \\ S_1, & S_2, & S_3 \dots (S) \end{vmatrix},$$

this determinant, expressed as a function of $x, y, z \dots t$, is what, in the paper referred to, I called a secondary derivee, but which for the future I shall cite by the more concise and expressive name of a *connective* of the system of functions $F, G, H \dots K$ from which it is obtained. One prevailing principle regulates all the cases treated of in this and the antecedent memoir, namely that of forming linearly independent systems of augmentatives or connectives, or both, of the given system whose resultant is to be found, of the same degree one with the other, and equal in number (when this admits of being done) to the number of distinct terms in the functions thus formed. The resultant of these functions, treated as linear functions of the several combinations of powers of the variables in each term, will then be the resultant of the given system clear of all irrelevant factors. If the number of terms to be eliminated exceed the number of the functions, the elimination of course cannot be executed. If the contrary be the case, but the equality is restored by the rejection of a certain number of the equations, the resultant so obtained will vary according to the choice of the equations retained for the purpose of the elimination. The true resultant will not then coincide with any of the resultants so obtained, but will enter as a common factor into them all.

The following simple arithmetical principles will be found applicable and useful for quotation in the sequel:—

(a) The number of terms in a homogeneous function of p letters of the mth degree is

$$\frac{m(m+1)\dots(m+p-1)}{1 \cdot 2 \dots p}.$$

(*b*) The number of augmentatives of the $(m+n)$th degree belonging to a function of p letters of the mth degree is

$$\frac{(n+1)(n+2)\dots(n+p-1)}{1.2\dots p}.$$

(*c*) The number of solutions in integers (excluding zeros) of the equation $a_1+a_2+\dots+a_p=k$ is

$$\frac{(k-1)(k-2)\dots(k-p+1)}{1.2\dots(p-1)}.$$

To begin with the case of binary aggregatives. Let

$$\left.\begin{array}{l} F_m(x,y)+F_{m-\iota}(x,y)\lambda^\iota+F_{m-\iota'}(x,y)\mu^{\iota'}+\&\text{c.}\dots+F_{m-(\iota)}(x,y)\theta^{(\iota)} \\ G_n(x,y)+G_{n-\iota}(x,y)\lambda^\iota+G_{n-\iota'}(x,y)\mu^{\iota'}+\&\text{c.}\dots+G_{n-(\iota)}(x,y)\theta^{(\iota)} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ K_p(x,y)+K_{p-\iota}(x,y)\lambda^\iota+K_{p-\iota'}(x,y)\mu^{\iota'}+\&\text{c.}\dots+K_{p-(\iota)}(x,y)\theta^{(\iota)} \end{array}\right\} \quad (A)$$

be a system of functions (whose Resultant it is proposed to determine) equal in number to the variables $x, y, \lambda, \mu \dots \theta$, and similarly aggregative, that is having only the same powers of λ, μ, &c. entering into them, but of any degrees equal or unequal $m, n \dots p$. Let the number of the functions be r. Raise each of the given functions by augmentation to the degree s, where

$$s=(m+n+\dots+p)-(\iota+\iota'+\dots+(\iota))-1,$$

the number of augmentatives of the several functions will be

$$(s+1)-m,$$
$$(s+1)-n,$$
$$\dots\dots\dots\dots$$
$$(s+1)-p,$$

and the total number will therefore be

$$r(s+1)-(m+n+\dots+p),$$

which $$=(r-1)(m+n+\dots+p)-r(\iota+\iota'+\dots+(\iota)).$$

Again, the number of terms to be eliminated will be the sum of the numbers of terms in functions respectively of the sth, $(s-\iota)$th, $(s-\iota')$th, ... $(s-(\iota))$th degrees, which are respectively

$$s+1,$$
$$s+1-\iota,$$
$$s+1-\iota',$$
$$\dots\dots\dots\dots$$
$$s+1-(\iota),$$

and the number of these partial functions is $r-1$. Hence the number of terms to be eliminated is

$$(r-1)\{m+n+\&c.+p-(\iota+\iota'+\&c.+(\iota))\}-(\iota+\iota'+\&c.+(\iota))$$
$$=(r-1)(m+n+\&c.+p)-r(\iota+\iota'+\ldots+(\iota)),$$

which is exactly equal to the number of the augmentative functions. Hence the Resultant* of the given functions can be found dialytically by linear elimination, and the exponent of its dimensions in respect to the coefficients of the given functions will be the number

$$(r-1)\,\Sigma m - r\Sigma\iota,$$

as above found.

The method above given may be replaced by another more compendious, and analogous to that known by the name of Bezout's abridged method for ordinary functions of two letters. As the method is precisely the same whatever the number of the functions employed may be, I shall for the sake of greater simplicity restrict the demonstration to the case of three functions, U, V, W, whose degrees (if unequal, written in ascending order of magnitude) are m, n, p respectively. Let

$$U = F_m(x, y) + F_{m-\iota}(x, y)\,z^\iota,$$
$$V = G_n(x, y) + G_{n-\iota}(x, y)\,z^\iota,$$
$$W = H_p(x, y) + H_{p-\iota}(x, y)\,z^\iota.$$

Let θ, ω be taken any two numbers which satisfy in integers greater than zero the equation $\theta+\omega=m+1$, and let

$$F_m(x, y) = \phi_{m-\theta}\cdot x^\theta + \phi_{m-\omega}\cdot y^\omega,$$
$$G_n(x, y) = \gamma_{n-\theta}\cdot x^\theta + \gamma_{n-\omega}\cdot y^\omega,$$
$$H_p(x, y) = \eta_{p-\theta}\cdot x^\theta + \eta_{p-\omega}\cdot y^\omega,$$

where the ϕ's, γ's, η's may be always considered rational integer functions of x and y; for every term in each of the functions F_m, G_n, H_p must either contain x^θ or y^ω, since, if not, its dimensions in x and y would not exceed

$$(\theta-1)+(\omega-1),$$

that is $m-1$, whereas each term is of m conjoined dimensions, at least, in x and y. Hence from the equations

$$U=0,$$
$$V=0,$$
$$W=0,$$

* The Resultant of a system of functions means in general the same thing as the left-hand side of the final equation (clear of extraneous factors) resulting from the elimination of the variables between the equations formed by equating the said functions severally to zero.

by eliminating x^{ω}, y^{θ} and z^{ι} we obtain the connective determinant

$$\begin{vmatrix} \phi_{m-\theta}, & \phi_{m-\omega}, & F_{m-\iota} \\ \gamma_{n-\theta}, & \gamma_{n-\omega}, & G_{n-\iota} \\ \eta_{p-\theta}, & \eta_{p-\omega}, & H_{p-\iota} \end{vmatrix},$$

which will be of the degree

$$m+n+p-(\theta+\omega+\iota),$$

that is of the degree $(n+p-\iota-1)$ in x and y; and the number of such connectives by principle (c) is p.

Again, by augmentation we can raise each of the functions U, V, W to the same degree as the connectives, and by principle (b) the number of such will be

$$n+p-m-\iota,$$

$$p-\iota,$$

$$n-\iota,$$

from U, V, W respectively, together making up the number

$$2n+2p-m-3\iota.$$

Hence in all we have $2n+2p-3\iota$ equations; and the number of terms to be eliminated will be, $n+p-\iota$ arising from F_m, G_n, H_p, and $n+p-2\iota$ from $F_{m-\iota}$, $G_{n-\iota}$, $H_{p-\iota}$; together making up the proper number $2n+2p-3\iota$.

Each connective contains ternary combinations of the coefficients, namely one of the coefficients belonging to that part of U, V, W which contains z^{ι}, and two coefficients from the other part: the dimensions of the resultant in respect of the coefficients of the former will hence be readily seen to be equal to the number of connectives + the number of terms in the augmentatives into which z^{ι} enters, that is, will equal $m+n+p-2\iota$; the total dimensions of the resultant in respect to all the coefficients of U, V, W will be

$$3m+(2n+2p-m-3\iota),$$

that is, $$2m+2n+2p-3\iota;$$

and consequently, in respect to the coefficients of F_m; G_n; H_p, will be of

$$(2m+2n+2p-3\iota)-(m+n+p-2\iota),$$

that is, of $m+n+p-\iota$ dimensions. This result, which is of considerable importance, may be generalized as follows.

Returning to the general system (A), for which we have proved that the total dimensions of the resultant are

$$(r-1)(m+n+\ldots+p)-r(\iota+\iota'+\ldots+(\iota)),$$

let the coefficients of the column of partial functions

$$F_m,\quad G_n,\quad \vdots\quad K_p,$$

be called the first set; the coefficients of the column

$$F_{m-\iota},\quad G_{n-\iota},\quad \vdots\quad K_{p-\iota},$$

the second set, and so forth; then the dimensions in respect of the 1st, 2nd ... $(r-1)$th sets respectively are s, $s-\iota$, $s-\iota'$ ... $s-(\iota)$, where

$$s = m + n + \&c. + p - (\iota + \iota' + \&c. + (\iota)).$$

The important observation remains to be made, that all the above results remain good although any one or more of the indices of dimension of the partial functions in the system (A), as $m-\iota$, $m-\iota'$, $n-\iota$, &c., should become negative, provided that the terms in which such negative indices occur be taken zero, as will be apparent on reviewing the processes already indicated upon this supposition. If we take

$$m = n = \ldots = p, \text{ and } \iota = \iota' = \&c. = (\iota) = m - \epsilon,$$

the exponent of the total dimensions of the resultant becomes

$$(r-1)\,rm - r\,(r-2)\,(m-\epsilon)$$
$$= rm + r\,(r-2)\,\epsilon,$$

when $\epsilon = 0$, this becomes mr, which is made up of $2m$ units of dimension belonging to the coefficients of the first column, and of m belonging to each of the $(r-2)$ remaining columns. Consequently, if we have

$$F_m(x,\ y) + \xi\lambda + \xi'\lambda' = 0,$$
$$G_m(x,\ y) + \eta\lambda + \eta'\lambda' = 0,$$
$$H_m(x,\ y) + \zeta\lambda + \zeta'\lambda' = 0,$$
$$K_m(x,\ y) + \theta\lambda + \theta'\lambda' = 0,$$

or any other number of equations similarly formed, the result of the elimination is always of m dimensions only in respect of ξ, η, ζ, θ, or of ξ', η', ζ', θ', and of $2m$ in respect of the coefficients in F, G, H, K.

I now proceed to state and to explain some seeming paradoxes connected with the degree of the resultant of such systems of defective functions as have been previously treated of in this memoir, as compared with the degree of the general resultant of a corresponding system of *complete* functions of the same number of variables.

In order to fix our ideas, let us take a system of only three equations of the form

$$\left.\begin{array}{l} F_m(x,\ y) + F_{m-\iota}(x,\ y)\, z^\iota = 0 \\ G_n(x,\ y) + G_{n-\iota}(x,\ y)\, z^\iota = 0 \\ H_p(x,\ y) + H_{p-\iota}(x,\ y)\, z^\iota = 0 \end{array}\right\} . \qquad \text{(B)}$$

The resultant of this system found by the preceding method is in all of $2m + 2n + 2p - 3\iota$ dimensions. But in general, the resultant of three equations of the degrees m, n, p is of $mn + mp + np$ dimensions.

Now in order to reason firmly and validly upon the doctrine of elimination, nothing is so necessary as to have a clear and precise notion, never to be let go from the mind's grasp, of the proposition that every system of n homogeneous functions of n variables has a single and invariable Resultant. The meaning of this proposition is, that a function of the coefficients of the given functions can be found, such that, *whenever* it becomes zero, and *never except* when it becomes zero, the functions may be simultaneously made zero for some certain system of ratios between the variables. The function so found, which is sufficient and necessary to condition the possibility of the coexistence of the equality to zero of each of the given functions, is their resultant, and by analogy they may be termed its components. It follows that if R be a resultant of a given system of functions, any numerical multiple of any power of R or of any root of R when (upon certain relations being supposed to be instituted between the coefficients of its components) R breaks up into equal factors, will also be a resultant. This is just what happens in system (B) when $m = n = p = \iota$; the resultant found by the method in the text is of the degree $3m$; the general resultant of the system of three equations to which it belongs is of the degree $3m^2$; the fact being, that the latter resultant becomes a perfect mth power for the particular values of the coefficients which cause its components to take the form of the functions in system (B).

Suppose, however, that we have still $m = n = p$, but ι less than m, $6m - 3\iota$ will express the degree of the resultant of system (B); but this is no longer in general an aliquot part of $3m^2$, and consequently the resultant of system (B) that we have found is no longer capable in general of being a root of the general resultant. The truth is, that on this supposition the general resultant is zero; as it evidently should be, because the values $\frac{x}{z} = 0$, $\frac{y}{z} = 0$ satisfy the equations in system (B), except for the case of $m = \iota$; consequently the resultant furnished in the text, although found by the same process, is something of a different nature from an ordinary resultant; it expresses, not that the system of equations (B) may be capable of coexisting, but that they may be capable of coexisting for values of $\frac{x}{z}, \frac{y}{z}$ other than 0 and 0. This is what I have elsewhere termed a sub-resultant. But there is yet a further case, to which neither of the above considerations will apply. This is when m, n, p are not equal, but $p - \iota = 0$.

On this supposition the degree of the resultant of (B) becomes $2m+2n-p$, which in general will not be a factor of $mn + mp + np$; and in this case it will no longer be true that the values $\frac{x}{z} = 0$, $\frac{y}{z} = 0$ will satisfy the system (B), inasmuch as the last equation therein cannot so be satisfied. Now, calling the general resultant R and the particular resultant R', if R' should break up into factors so as to become equal to $(r')^a \times (s')^b \ldots (t')^c$, it might be the case that R should equal $(r')^\alpha . (s')^\beta \ldots (t')^\gamma$, and there would be nothing in this fact which would be inconsistent with the theory of the resultant as above set forth; but suppose that R' is indecomposable into factors, then it is evident that we must have $R = R' . R''$, and consequently that the existence of such a particular resultant as R' will argue the necessity of the existence of another resultant R''; in other words, the resultant so found cannot be in a strict sense the true and complete resultant for the particular case assumed, and yet the process employed appears to give the complete resultant, or at least it is difficult to see how the wanting factor escapes detection. To make this matter more clear, take a particular and a very simple case, where $m = 2$, $n = 2$, $p = \iota = 1$, so as to form the system of equations

$$\left.\begin{array}{l} Ax^2 + Bxy + Cy^2 + (Dx + Ey)\, z = 0 \\ A'x^2 + B'xy + C'y^2 + (D'x + E'y)\, z = 0 \\ lx + my + nz \qquad\qquad = 0 \end{array}\right\}. \qquad \text{(C)}$$

By virtue of my theorem, the degree of the resultant R' is

$$2\,(2 + 2 + 1) - 3\,.\,1 = 7,$$

but the resultant R of the system

$$\left.\begin{array}{l} Ax^2 + Bxy + Cy^2 + (Dx + Ey)\, z + Fz^2 = 0 \\ A'x^2 + B'xy + C'y^2 + (D'x + E'y)\, z + F'z^2 = 0 \\ lx + my + nz \qquad\qquad = 0 \end{array}\right\}, \qquad \text{(D)}$$

which becomes identical with the former when $F = 0$, $F' = 0$ is of

$$2 \times 2 + 2 \times 1 + 2 \times 1,$$

that is, of 8 dimensions. Hence it is evident that when $F = 0$, $F' = 0$, R must become $R' \times R''$.

It will be found in fact*, that on the supposition of $F=0, F'=0$, R becomes equal to $N \times R'$; and accordingly, besides the portion R' of the resultant of system (C), found by the method in the text, there is another portion N which has dropped through; but it may be asked, is N truly a relevant factor? were it not so, the theory of the resultant would be completely invalidated; but in truth *it is*; for $N=0$ will make the equations in system (C), *considered as a particular case* of system (D), capable of co-existing; the peculiarity, which at first sight prevents this from being obvious, consisting in the fact that the values of $\frac{x}{z}, \frac{y}{z}$ which satisfy the three equations when $N=0$ become *infinite*.

Thus, finally, we have arrived at a clear and complete view of the relation of the particular to the general resultant.

The general resultant may be zero, in which case the particular resultant is something altogether different from an ordinary resultant; or the particular resultant may be a root of the general resultant, or it may be more generally the product of powers of the simple factors, which enter into the composition of the general resultant; or lastly, it may be an incomplete resultant, the factors wanting to make it complete being such as when equated to zero, will enable the components of the resultant to coexist, but not for other than infinite values of certain of the ratios existing between the variables.

Without for the present further enlarging on the hitherto unexplored and highly interesting theory of Particular Resultants, I will content myself with stating one beautiful and general theorem relating to them; to wit, "if $F=0$, $G=0$, &c. be a given system of equations with the coefficients left general, and R be the resultant of F, G, &c., and if now the coefficients in F, G be so taken that R comes to contain as a factor or be coincident with R'^m, then will $R'=0$ indicate that (when the coefficients are so taken as above supposed) $F=0$, $G=0$, &c. will be capable of being satisfied, not, as in general, by one only, but by m distinct systems of values of the variables in F, G, &c., subject of course to the possibility, in special cases, of certain of the systems becoming multiple coincident systems."

I pass on now* to the more recondite and interesting theory of the resultant of Ternary Aggregative Functions, that is to say, functions of the form

$$F_m(x, y, z) + F_{m-\iota}(x, y, z)\,t^{\iota} + \&c. \ldots + F_{m-(\iota)}(x, y, z)\,t^{(\iota)},$$

which will be seen to admit of some remarkable applications to the theory of reciprocal polars.

[* See the Author's remarks below, p. 283.]