42.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

[*Cambridge and Dublin Mathematical Journal*, VII. (1852), pp. 52—97.]

PART I. GENERATION OF FORMS*.

SECTION I. *On Simple Concomitance.*

THE primary object of the Calculus of Forms is the determination of the properties of Rational Integral Homogeneous Functions or systems of functions: this is effected by means of transformation; but to effect such transformation experience has shown that forms or form-systems must be contemplated not merely as they are in themselves, but with reference to the ensemble of forms capable of being derived from them, and which constitute as it were an unseen atmosphere around them. The first part of this essay will therefore be devoted to the theory of the external relations of forms or form-systems; the second part to the analysis of forms: that is to say, the first part will treat of the Generation and affinities, and the second part of the Reduction and equivalences of forms.

In its most crude and absolute, or, so to speak, archetypal condition a Rational Integral Homogeneous Function may be regarded as a linear function of several distinct and perfectly independent classes of variables.

* It may be well at the outset to give notice to my readers of the exact meaning to be attached to the following terms:

1. The linear-transformations are supposed to be always taken such that the modulus, that is, the determinant of the coefficients of transformation, is unity; or, as it may be phrased, the transformations are uni-modular.

2. The word Determinant is restricted in all cases to signify the alternate function formed in the usual manner from a group of quantities arranged in square order.

3. The word *Discriminant* (typified by the prefix-symbol □) is used to denote the determinant (usually but most perplexingly so called) of a homogeneous function of variables.

4. The resultant of two or more homogeneous functions of as many variables is the left-hand side of the final equation (in its *complete form* and free from extraneous factors) which results from eliminating the variables between the equations obtained by making each of the functions zero.

The first step towards the limitation of this very general but necessary conception consists in imagining the total number of classes to become segregated into groups, and certain correspondences to obtain between the variables of a class in any group with some the variables in each other class of the same group. The investigations in this and the subsequent section will be confined exclusively to the theory of functions where the several classes of variables, if more than one, all belong to a single group, so that the variables in one class have each their respective correspondents in the remaining classes. Such a group may again be conceived to become subdivided into sets each of the same number of variables, and the corresponding variables in the different sets to become absolutely identical. This leads to the conception of a homogeneous function of related classes of variables of various degrees of exponency in respect to the several classes. The relation of the different classes, if containing the same number of variables (in which case the relation may be termed Simple) will be understood to be defined by their being simultaneously subject to similar or contrary operations of linear substitution; so that, for example, if x, y, z; ξ, η, ζ are two such classes, when x, y, z are replaced by $ax+by+cz$, $a'x+b'y+c'z$, $a''x+b''y+c''z$, respectively, ξ, η, ζ will be, according to the *species* of the relation, subject to be at the same time replaced either by $a\xi+b\eta+c\zeta$, $a'\xi+b'\eta+c'\zeta$, $a''\xi+b''\eta+c''\zeta$, or otherwise by $\alpha\xi+\beta\eta+\gamma\zeta$, $\alpha'\xi+\beta'\eta+\gamma'\zeta$, $\alpha''\xi+\beta''\eta+\gamma''\zeta$, where

$$\alpha = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b' & c' \\ 0 & b'' & c'' \end{vmatrix} \qquad \beta = \begin{vmatrix} 0 & 1 & 0 \\ a' & 0 & c' \\ a'' & 0 & c'' \end{vmatrix} \qquad \gamma = \begin{vmatrix} 0 & 0 & 1 \\ a' & b' & 0 \\ a'' & b'' & 0 \end{vmatrix}$$

&c. &c. &c.*

On the former supposition the related classes x, y, z, ξ, η, ζ will be said to be cogredient, and on the latter supposition contragredient †. If now we have one or more functions of classes of variables so related ‡, such function or system of functions may have associated with it a concomitant, also made up of distinct but related classes of variables, such classes being capable of being either greater or fewer in number than the classes of the given function or system of functions.

In the primitive function or system, as also in the concomitant, the related classes may be all of the same species, or some of one and the others of the contrary species. Even if we limit ourselves to the conception of a

* See my paper in the previous number of this *Journal* [p. 199 above.]

† The germ of the notion of contragredience will be found in the immortal *Arithmetic* of the great and venerable Gauss.

‡ The relation here spoken of will be observed to be of a *dynamical* character, not referring to the systems as they are in themselves, but to the movements to which they are simultaneously subject.

primitive function or system of functions with only one class of variables, its concomitant may be composed of various classes of variables, in respect to some of which it will be covariant with, and in respect to the others contravariant to, the primitive function or system*. This is an immense and most important extension of the conception of a concomitant given in my preceding paper in this *Journal*, and will be shown to have the effect of reducing the whole existing theory under subjection to certain simple abstract and universal laws of operation.

The relation of concomitance is purely of form. A being a given form, B is its concomitant, when A' being derived from A by simultaneous substitutions impressed upon the class of variables or upon each of the classes (if there be more than one) in A, and B' from B by corresponding (coincident or contrary) substitutions impressed upon the class or classes of variables in B, B' is capable of being derived from A' after the same law as B from A; or, as it may be otherwise expressed, "functions are concomitant when their correlated linear derivatives are homogeneous in point of form †."

This definition implies that one at least of the forms must be the most general possible of its kind: in a secondary but very important sense, however, functions obtained by impressing particular values or relations upon the quantities entering into the primitive and its associate form, will still be called concomitant. Thus $x^3 - y^3$ will be termed a concomitant to $x^3 + y^3$, not that we can affirm that $(ax + by)^3 - (cx + dy)^3$:

that is $\qquad (a^3 - c^3)\,x^3 + 3\,(a^2b - c^2d)\,x^2y + 3\,(ab^2 - cd^2)\,xy^2 + (b^3 - d^3)\,y^3,$

treated as a function of x and y, can be derived from $(ax + by)^3 + (cx + dy)^3$,

that is $\qquad (a^3 + c^3)\,x^3 + 3\,(a^2b + c^2d)\,x^2y + 3\,(ab^2 + cd^2)\,xy^2 + (b^3 + d^3)\,y^3,$

when $ad - bc = 1$ by the same law as $(x^3 - y^3)$ from $(x^3 + y^3)$, for the elements for forming such comparison are wanting, but because $x^3 + y^3$ and $x^3 - y^3$ are the correspondent particular values respectively assumed by

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

and its concomitant

$$(ad^2 + 2c^3 - 3bcd)\,x^3 - (6b^2d - 3c^2b - 3acd)\,x^2y$$
$$+ (6ac^2 - 3cb^2 - 3cba)\,xy^2 - (a^2d + 2b^3 - 3bca)\,y^3,$$

when $\qquad a = 1, \quad b = 0, \quad c = 0, \quad d = 1.$

With the aid of this extended signification of the term concomitant (whether it be a covariant or contravariant) we can in all cases speak (as otherwise we in general could not) of the concomitant of a concomitant. The relation

* And of course the concomitant may be an invariant to its originant in respect of one or more systems of variables entering into the former.

† Or, more generally, it may be said that concomitance consists in the persistence of morphological affinity.

between systems of variables has been stated to be Simple (whether they be cogredient or contragredient) when each variable in one system corresponds with some one in each other. Compound relation arises as follows:—Suppose $x, y; \xi, \eta$ two independent systems of two variables each, and that the system of four variables u, v, w, t is subject to linear variations imitating, in the way of cogredience or contragredience, those to which $x\xi, x\eta, y\xi, y\eta$ are subject; then u, v, w, t may be said to be cogredient or contragredient to the continued systems $x, y; \xi, \eta$. If $x, y; \xi, \eta$ be themselves cogredient, then a system of only three variables u, v, w, may be cogredient or contragredient in respect to $x\xi, x\eta + y\xi, y\eta$, and if $x, y; \xi, \eta$ be coincident, u, v, w may be similarly related to x^2, xy, y^2. The illustration may easily be generalized, and it will be seen in the sequel that its conception of compound-relation between systems of a differing number of variables will greatly extend the power and application of the methods about to be developed. Without having recourse to a formal definition, it is obvious that the notion of a concomitant conveyed in my former paper in this *Journal* lends itself without difficulty to the most general supposition which can be made of functions between which any number of systems of related variables are distributed, whatever such relation be, whether simple or compound, and whether of cogredience or of contragredience. The proposition stated in my last paper relative to a concomitant of the concomitant of a function being a concomitant of the original still applies to concomitants in the wider sense in which we now understand that term, and the species of each system of variables in the second concomitant with respect to the species or either species (if there be systems of both kinds in the primitive) will be determined upon the general principle which determines the effect of concurrence and contrariety being made to operate each upon itself or one in either order upon the other.

The highest law and the most powerful in its applications which I have yet discovered in the theory of concomitants may be expressed by affirming that when several related classes of variables are present in any concomitant, a new concomitant, *derived from the former by treating one or any number of these classes as independent of the remaining classes*, will still be a concomitant of the primitive. I shall quote this hereafter as the Law of Succession. This law, to which I have been led up inductively, requires an extended examination and a rigorous proof. It is the keystone of the subject, and any one who should suppose that it is a self-evident proposition (as from the simplicity of the enunciation it might be supposed to be) will commit no slight error.

If $\phi(x, y \ldots z)$ be any homogeneous form of function of $x, y, \ldots z$, every homogeneous sum in the expansion by Taylor's theorem of

$$\phi(u + u', \ v + v' \ldots w + w'),$$

which in fact, on making $u' = x$, $v' = y \ldots w' = z$, becomes identical (to a numerical factor *près*) with $\left(u \frac{d}{dx} + v \frac{d}{dy} + w \frac{d}{dz}\right)^{\iota} \phi$, is what I have elsewhere termed an Emanant, and by a partial method I had demonstrated that every invariant of such an emanant in respect to $u, v \ldots w$, in which $x, y \ldots z$ are treated as constants, or *vice versâ*, would give a covariant of ϕ. The reason of this is now apparent. For it may easily be shown* that every emanant is in fact itself a covariant of the function to which it belongs with respect to each of the related classes of variables which enter into it, or is as it may be termed a double covariant. The law of Succession shows therefore that a concomitant to an emanant from which one of the classes has disappeared will be a covariant of the primitive in respect to the remaining class.

In applying the law of Succession, great use can be made of a function of two classes of letters which may be termed a Universal Mixed Concomitant; this is $x\xi + y\eta + \ldots + z\zeta$, which has the property of remaining unaltered when any linear substitution (for which the modulus is unity) is impressed upon $x, y \ldots z$, and the contrary one upon $\xi, \eta \ldots \zeta$†.

If $f(x, y)$ be any function of x, y, of the degree m, $f + \lambda(x\xi + y\eta)^m$ will

* To demonstrate this it is only necessary to observe that if $u, v, \ldots w, u', v', \ldots w'$ be cogredient with themselves and with $x, y, \ldots z$,

$$\phi(u + \lambda u', \ v + \lambda v', \ \ldots \ w + \lambda w')$$

will evidently be a concomitant of $\phi(x, y, \ldots z)$; and, λ being arbitrary, the coefficients of the different powers of λ must be separately concomitants of $\phi(x, y, \ldots z)$, but these coefficients are the emanants of ϕ. Q. E. D.

† Thus, if

$$\begin{aligned}
x &= ax' + by' + cz', & \xi &= (gn - hm)\,\xi' + (hl - fn)\,\eta' + (fm - gl)\,\zeta', \\
y &= fx' + gy' + hz', & \eta &= (-nb + mc)\,\xi' + (-lc + na)\,\eta' + (-ma + lb)\,\zeta', \\
z &= lx' + my' + nz', & \zeta &= (bh - cg)\,\xi' + (cf - ah)\,\eta' + (ag - bf)\,\zeta',
\end{aligned}$$

then
$$x\xi + y\eta + z\zeta = \begin{pmatrix} a & b & c \\ f & g & h \\ l & m & n \end{pmatrix} \times (x'\xi' + y'\eta' + z'\zeta')$$
$$= x'\xi' + y'\eta' + z'\zeta'.$$

When the coefficients of transformation correspond to the direction-cosines between one system of rectangular axes and another, the reciprocal system is identical with the direct system; so that x, y, z; ξ, η, ζ, on this particular supposition, may be regarded indifferently as contragredient or as cogredient; accordingly they may be made identical, and then $x^2 + y^2 + z^2$ remains invariable, which is the well-known characteristic of orthogonal transformation. It may be observed here that there exists a special theory of concomitance limited to such species of linear transformations, which may be termed Conditional Concomitance, and I have found in several cases that the invariants of conditional concomitants turn out to be absolute invariants of the primitive. Much more important is the remark that there exists a theory of universal concomitants for an indefinite number instead of merely two systems of variables, as used in the text. In the sequel it will be seen that the application of this universal concomitant (like the touch of an enchanter's wand) serves to transmute covariants into contravariants, and back again, and causes single invariants to germinate and fructify into complete connected systems of forms.

be a mixed concomitant of f, it being evident that every function of concomitants of a function is itself a concomitant of the same.

Suppose now

$$f = ax^m + mbx^{m-1}y + \tfrac{1}{2}m(m-1)cx^{m-2}y^2 + \&c.,$$

the concomitant becomes

$$(a+\lambda\xi^m)x^m + m(b+\lambda\xi^{m-1}\eta)x^{m-1}y + \tfrac{1}{2}m(m-1)(c+\lambda\xi^{m-2}\eta^2) + \&c.$$

Consequently if P be any concomitant of f, P' obtained from P by writing $a+\lambda\xi^m$, $b+\lambda\xi^{m-1}\eta$, &c. for a, b, &c., will still be a concomitant of f; and by Taylor's theorem P' evidently equals

$$P + \left(\xi^m\frac{d}{da} + \xi^{m-1}\eta\frac{d}{db} + \&c.\right)P$$
$$+\frac{1}{1.2}\left(\xi^m\frac{d}{da} + \xi^{m-1}\eta\frac{d}{db} + \&c.\right)^2 P$$
$$+ \&c.$$

If we take P an invariant of f, we have M. Hermite's theorem* for $f(x, y)$, and precisely the same demonstration applies to the general case of $f(x, y \dots z)$. P' is, by virtue of the general rule, a contravariant of f in respect to $\xi, \eta \dots \zeta$: if P be taken a function containing one single system, and is also a contravariant to f in respect to that system, P' will be a double contravariant; and if we make the two systems in P' identical, we have the extension of M. Hermite's theorem alluded to by me in one of the notes† to my last paper, wherein I have stated that "I may be taken any *covariant* of the function": as regards the purpose of that statement, the word covariant was used in error for contravariant.

The preceding method may be viewed as a particular application of the general principle, that if $U_1, U_2 \dots U_m$ be any m functions (whether concomitants any of them of the others or not), then any concomitant of $\lambda_1U_1 + \lambda_2U_2 + \dots + \lambda_mU_m$ being expressed as a function of $\lambda_1, \lambda_2 \dots \lambda_m$, every coefficient in such expression will be a concomitant of the system $U_1, U_2 \dots U_m$. Thus, for example, if U and V be two quadratic functions of n variables $x, y \dots z$, the *discriminant* $\square(\lambda U + \mu V)$ will contain $n+1$ terms, of which the coefficients of the first and last will be $\square U$ and $\square V$; and every one of the $(n+1)$ coefficients will be a concomitant (of course an invariant) of U and V. These $(n+1)$ invariants will in fact constitute the fundamental scale of invariants to the system U and V, and every other invariant of U

* This theorem was first stated to me by Mr Cayley, who, I understand, derived it from M. Eisenstein, under the form of a theorem of covariants, which of course it becomes on interchanging x, y with $-y, x$. But as a theorem of covariants it could not be extended to functions of more than two variables. M. Hermite appears to have discovered this theorem, under its more eligible form, subsequently to, but independently of, M. Eisenstein.

[† p. 201 above, note *.]

and V will be an explicit rational function of the $(n+1)$ terms of the scale. In connexion with this principle may be stated another relative to any system of homogeneous functions of a greater number of variables of the same class, namely, that if any set of the variables one less in number than the number of the functions be selected at will, and any invariant of a given kind be taken of the resultant of the functions in respect to the variables selected, all such invariants so formed will have an integral factor in common, and this common factor will be an invariant of the given system of functions.

It will be convenient to speak hereafter of systems for which the march of the linear substitutions is coincident as cogredient, and those for which the march is contrary as contragredient systems.

Suppose m cogredient classes of m variables, the determinant formed by writing the $m \times m$ quantities in square order will evidently be a universal covariant. Thus, take the two systems x, y; ξ, η. $x\eta - y\xi$ is a universal covariant, and evidently therefore F, which I use to denote

$$\phi(x, y) \times \phi(\xi, \eta) + \lambda(x\eta - y\xi)^m,$$

will be a covariant to $\phi(x, y)$. Let $\phi(x, y)$ be of m dimensions; any invariant of F will be an invariant of ϕ; thus, let the two systems x, y; ξ, η be treated as perfectly independent, and take the discriminant of F (viewed as a function of x, y; ξ, η), that is the resultant of the four functions $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{d\xi}, \frac{dF}{d\eta}$; this resultant will be an invariant of ϕ; and λ being arbitrary, all the coefficients of its different powers will be invariants of ϕ. We thus fall upon another theorem of M. Hermite, namely that if $\lambda = \frac{\phi(x, y) \times \phi(\xi, \eta)}{(x\xi - y\eta)^m}$, the coefficients of the equation which will give the minimum values of λ are invariants of ϕ. So more generally, any invariant of $f(x, y, \xi, \eta) - \lambda(x\xi - y\eta)^m$, f being of the degree m in x, y and in ξ, η, will be an invariant of f; and among other invariants may be taken the discriminant obtained by treating x, ξ, y, η as absolutely unrelated.

If f be a function of various classes each containing n covariables, and if not less than n of these classes be covariable classes, and after selecting at will any n of such systems, as $x_1, y_1 \ldots z_1$; $x_2, y_2 \ldots z_2$; ...... $x_n, y_n \ldots z_n$, the symbolical determinant

$$\begin{vmatrix} \frac{d}{dx_1}, & \frac{d}{dy_1} & \cdots & \frac{d}{dz_1} \\ \frac{d}{dx_2}, & \frac{d}{dy_2} & \cdots & \frac{d}{dz_2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d}{dx_n}, & \frac{d}{dy_n} & \cdots & \frac{d}{dz_n} \end{vmatrix}$$

be expanded and written equal to D, then $D^\iota f$ will be a concomitant of f; and, more generally, by selecting different combinations of the covariable systems n and n together in every way possible, and forming corresponding symbols of operation $E, F \dots H$, we shall have $D^\iota . E^{\iota'} \dots H^{(\iota)} . f$, for all values of $\iota, \iota' \dots (\iota)$, a covariant of f in respect to the classes so combined. This explains and contains the whole pith and marrow of Mr Cayley's simple but admirable method of obtaining covariants and invariants (or, as termed by their author, hyperdeterminants) to a function ϕ_1 of a single system $x_1, y_1 \dots z_1$; he forms similar functions $\phi_2 \dots \phi_\mu$ of $x_2, y_2 \dots z_2$; $\dots x_\mu, y_\mu \dots z_\mu$, and uses the product $\phi_1 \times \phi_2 \times \dots \times \phi_\mu$ as a function f of μ systems: the multiple covariant obtained by operating thereupon becomes a simple covariant on identifying the different classes of covariables introduced in the procedure.

SECTION II. *On Complex Concomitance.*

We have hitherto been engaged in considering only a particular case of concomitance, the true idea of which relates not to an individual associated form (as such), but to a complex of forms capable of degenerating into an individual form. Such a complex may be called a Plexus. A plexus of forms is concomitant to a given form or combination of forms under the following circumstances.

If (O) be the originant, meaning thereby the primitive form or system of forms, and P the concomitant plexus made up of the μ forms $P_1, P_2 \dots P_\mu$, and if, when by duly related linear substitutions, O becomes O', the plexus P becomes P', made up of the forms $P'_1, P'_2 \dots P'_\mu$, and if the plexus $'P$ formed from O' after the same law as P from O be made up of the forms $'P_1, 'P_2 \dots 'P_\mu$, then will each form in either of the plexuses $'P, P'$ be a linear function of all the forms in the other plexus, and the connecting constants in every such linear function will be functions of the coefficients of the substitution whereby O and P have become transformed into O' and P'.

A function forming part of a concomitant plexus may be termed a concomitantive. Concomitantives therefore usually have a joint relation to a common plexus and a concomitant is only another name for an unique concomitantive. Every plexus contains a definite number of concomitantives; in place of any one of these may be substituted an arbitrary linear function of all the rest, but the total number of independent forms *sufficient and necessary* to make the complete plexus respond to the requirements of the definition will remain constant.

If now we combine together the whole number of functions contained in one or more plexuses concomitant to any given originant, all of the same degree relative to any given selected system or systems of variables, and if the number of the concomitantives so combined be exactly equal to the number of terms in each, arranged as a function of the selected class or classes of variables, then the dialytic resultant (obtained by treating each combination of the selected variables as an independent variable, and forming a determinant in the usual manner), will be a concomitant to the given originant. This, which is only the partial expansion of some much higher law, may be termed the "Law of Synthesis."

Let f be any function of a single class of variables $x_1, x_2 \dots x_n$. Let χ represent any product of these variables or of their several powers of any given degree r; the number of different values of χ will be μ, where

$$\mu = \frac{n(n+1)\dots(n+r-1)}{1.2\dots r},$$

and $\chi_1 f, \chi_2 f \dots \chi_\mu f$ will form a covariantive plexus to f.

Again, let ϑ represent any product of the degree r of the symbols

$$\frac{d}{dx_1}, \frac{d}{dx_2} \cdots \frac{d}{dx_n};$$

$\vartheta_1 f, \vartheta_2 f \dots \vartheta_\mu f$ will also form a covariant plexus to f.

The coefficients of connexion between the forms of either plexus depend in an analogous manner upon the coefficients of the substitution supposed to be impressed upon the variables, with the sole difference that every coefficient taken from the line r and column s of the determinant of substitution which appears in any coefficient of connexion of the one plexus is replaced by the coefficient taken from the line s and the column r in the corresponding coefficient of connexion for the other plexus.

Let $f(x, y)$ be any function of x, y of the degree $2m$; then

$$\left(\frac{d}{dx}\right)^m, \left(\frac{d}{dx}\right)^{m-1} \frac{d}{dy}, \dots\dots \left(\frac{d}{dy}\right)^m$$

will form a covariantive plexus; thus, suppose

$$f(x, y) = a_1 x^{2m} + 2ma_2 x^{2m-1} y + \dots + a_{2m+1} y^{2m};$$

omitting numerical factors, the plexus will be composed of the $(m+1)$ lines following:

$$\begin{array}{l} a_1 x^m \quad + ma_2 x^{m-1} y \quad + \dots + a_{m+1} y^m, \\ a_2 x^m \quad + ma_3 x^{m-1} y \quad + \dots + a_{m+2} y^m, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m+1} x^m + ma_{m+2} x^{m-1} y + \dots + a_{2m+1} y^m, \end{array}$$

and consequently, by the law of synthesis, the determinant

$$\begin{vmatrix} a_1, & a_2 \dots\dots a_{m+1} \\ a_2, & a_3 \dots\dots a_{m+2} \\ \dots\dots & \dots\dots\dots\dots \\ a_{m+1}, & a_{m+2} \dots a_{2m+1} \end{vmatrix}$$

is an invariant of f.

When this determinant is zero, I have proved in my paper* on Canonical Forms, in the *Philosophical Magazine* for November last, that f is resoluble into the sum of m powers of linear functions of x and y. I shall hereafter refer to a determinant formed in this manner from the coefficients of f as its catalecticant. Mr Cayley was, I believe, the first to observe that all catalecticants† are invariants.

Again, more generally, let $f(x, y, \xi, \eta)$ be a function of the mth degree of x, y, and of a like degree in respect of ξ, η, which are supposed to be cogredient with x and y; then

$$f(x, y, \xi, \eta) + \lambda(x\eta - y\xi)^m$$

(say F) will be a concomitant of f; and therefore if we take the system

$$\left(\frac{d}{dx}\right)^m F, \left(\frac{d}{dx}\right)^{m-1} \frac{d}{dy} F \ldots\ldots \left(\frac{d}{dy}\right)^m F,$$

which will be functions of ξ and η alone, and take their resultant, this resultant will be an invariant of f. As a particular case of this theorem, let

$$f = \left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^m \phi,$$

where ϕ is supposed to be a function of x and y only and of $2m$ dimensions, f is a concomitant of ϕ, and therefore the invariant of f, obtained in the manner just explained, will be an invariant of ϕ. Thus then we have an instantaneous demonstration of the theorem given‡ by me in the paper of the *Philosophical Magazine* before named, namely, if

$$\phi(x, y) = a_1 x^{2m} + 2ma_2 x^{2m-1} y + \ldots + a_{2m+1} y^{2m},$$

say, in order to fix the ideas, $= ax^6 + 6bx^5y + 15cx^4y^2 + \ldots + gy^6$; then the determinant

$$\begin{vmatrix} a, & b, & c, & d+\lambda \\ b, & c, & d-\frac{1}{3}\lambda, & e \\ c, & d+\frac{1}{3}\lambda, & e, & f \\ d-\lambda, & e, & f, & g \end{vmatrix},$$

(and the analogously formed determinant for the general case) will be an invariant of ϕ. The general determinant so formed is peculiarly interesting, because it furnishes when equated to zero the one sole equation necessary to be solved in order to be able to effect the reduction of $\phi(x, y)$ to its canonical form, and gives the means, irrespective of any other view of the theory of invariants, of determining completely and absolutely the condition

[* See p. 282 above.]

† But the catalecticant of the biquadratic function of x, y was first brought into notice as an invariant by Mr Boole; and the discriminant of the quadratic function of x, y is identical with its catalecticant, as also with its Hessian. Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant.

[‡ p. 277 above.]

of the possibility of two given functions of the same degree of x, y being linearly transformable one into the other. This theorem will be obtained in a more general manner in the following section. I only pause now to make the very important observation, that not only is the determinant an invariant, but every minor system* of determinants that can be formed from it (there are of course m such systems) is an invariantive plexus to the given function ϕ.

The form under which this theorem presents itself suggests a theorem vastly more general and of peculiar interest, as showing a connexion between the theory of functions of a certain degree and of a certain number of variables with other functions of a lower degree but of a greater number of variables. Here again, under a different aspect, is reproduced the great principle of dialysis, which, originally discovered in the theory of elimination, in one shape or another pervades the whole theory of concomitance and invariants.

Let ϕ represent any function of the degree pq (of any number, or, to fix the ideas, say of three variables x, y, z); let the general term of ϕ be represented by

$$\frac{pq\,(pq-1)\dots 1}{(1\,.\,2\dots\alpha)(1\,.\,2\dots\beta)(1\,.\,2\dots\gamma)}(\alpha, \beta, \gamma)\,x^\alpha y^\beta z^\gamma,$$

where $\alpha+\beta+\gamma=pq$, and (α, β, γ) represents a portion of the coefficient of $x^\alpha y^\beta z^\gamma$.

Let

$$\frac{1\,.\,2\dots p}{(1\,.\,2\dots r)(1\,.\,2\dots s)(1\,.\,2\dots t)}\,x^r y^s z^t = \theta_{r,\,s,\,t},$$

where $r+s+t=p$, so that there are as many θ's as there are modes of

* These minor systems mean as follows:—the system of rth minors comprises all the distinct determinants that can be got by striking out from the square array (which I call the Matrix) from which the complete determinant is formed, any r lines and any r columns selected at will. The last, or mth minor, is of course a system consisting of the coefficients of $\phi(x, y)$, and it is evident that if $\phi(x, y \dots z)$ be any function of any number of variables $x, y \dots z$, the coefficients will form an invariantive plexus to ϕ.

The following remark as to the changes undergone by the coefficients of ϕ when the variables undergo any substitution, is not without interest and importance for the theory.

Let x become $fx+f'y+\dots+(f)\,z$,

y .... $gx+g'y+\dots+(g)\,z$,

....

z .... $hx+h'y+\dots+(h)\,z$.

Then the coefficient of the highest power of x becomes

$$\phi(f, g \dots h),$$

and the coefficient of the term containing $y^r \dots z^s$ becomes

$$\left(f'\frac{d}{df}+g'\frac{d}{dg}+\dots+h'\frac{d}{dh}\right)^r \times \text{\&c.} \times \left\{(f)\frac{d}{df}+(g)\frac{d}{dg}+\dots+(h)\frac{d}{dh}\right\}^s \phi(f, g \dots h).$$

subdividing p into three integral parts (zeros being admissible); that is $\frac{1}{6}(p+1)(p+2)(p+3)$. Then any product such as $x^\alpha y^\beta z^\gamma$ may be divided in a variety of ways into the product of q of these θ's, and it may be shown that the entire quantity

$$\frac{pq(pq-1)\ldots 1}{(1.2\ldots\alpha)(1.2\ldots\beta)(1.2\ldots\gamma)}(x^\alpha y^\beta z^\gamma)$$

$$=\Sigma\left\{\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_r)}(\theta_{\mu_1}{}^{m_1}\theta_{\mu_2}{}^{m_2}\ldots\theta_{\mu_r}{}^{m_r})\right\},$$

where $m_1+m_2+\ldots+m_r=q$. Consequently ϕ may be represented under the form of a function of the degree q of $\frac{1}{6}(p+1)(p+2)(p+3)$ (say ι) variables $\theta_1, \theta_2\ldots\theta_\iota$, and its general term will be of the form

$$\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_r)}(\alpha, \beta, \gamma)\{\theta^{m_1}\theta^{m_2}\ldots\theta^{m_r}\},$$

where α, β, γ are the indices respectively of x, y, z, when the last factor is expressed as a function of these variables*. Now if ϑ be used to denote this new representation of ϕ when $\theta_1, \theta_2\ldots\theta_\iota$ are treated as absolutely independent variables, and if we attach to it any universal concomitant, as $(x\xi+y\eta+z\zeta)^p$ admitting of being written under the form $\omega(\theta_1, \theta_2\ldots\theta_\iota)$, wherein the coefficients will be functions of ξ, η, ζ; then any invariant to ϑ and ω, treated as two systems of ι variables, will be a concomitant to ϕ, the original function in x, y, z†. ϑ and ω may be termed respectively, for facility of reference, the Particular and Absolute functions. Thus, for example, we take ϕ a function of x, y of the degree $4n$, say

$$a_1x^{4n}+4na_2x^{4n-1}y+\&c.+a_{4n+1}y^{4n},$$

and make $p=2n$, $q=2$, so that ϑ becomes a quadratic function of $(2n+1)$ variables obtained by making $x^{2n}=\theta_1$, $x^{2n-1}y=\theta_2\ldots y^{2n}=\theta_{2n+1}$‡, and the concomitant ω, formed from $(\xi x+\eta y)^{2n}$, becomes

$$\theta_1\xi^{2n}+2n\theta_2\xi^{2n-1}\eta+\ldots+\theta_{2n+1}\eta^{2n};$$

then if we take R the quadratic invariant of ω, that is

$$R=\theta_1\theta_{2n+1}-2n\theta_2\theta_{2n}\ \&c.\pm\frac{1.2.3\ldots(2n)}{(1.2\ldots n)^2}\tfrac{1}{2}(\theta_{n+1})^2,$$

* See Note (1) in Appendix. [p. 322 below.]

† In fact ϑ is a concomitant to ϕ, and ω to a power of the universal concomitant; the θ's forming a system of variables cogredient with the compound system $x^{r_1}y^{s_1}z^{t_1}$, $x^{r_2}y^{s_2}z^{t_2}$, &c.: and it must be well observed that the same substitutions which render ϑ and ω respectively identical with ϕ and a power of the universal concomitant, would render an infinite number of other functions also coincident with the same; but none of these other functions would be concomitants. Herein we see the importance of the definition and conception of compound relation; the θ system being compound by relation with the x, y, z system, after the manner of cogredience.

‡ A slight variation upon the method as above explained for the general case has been here introduced inadvertently by writing $x^{2n-1}y=\theta_2$, &c., in lieu of $2nx^{2n-1}y=\theta_2$, &c., which, as it does not in any degree affect the reasoning, I have not deemed it worth while to alter.

it will readily be seen that the determinant of $\vartheta + \lambda R$, treated as a quadratic function of $(2n+1)$ variables, will give an invariant of ϕ, and this will be the same as that obtained by the particular method above given. Thus, suppose

$$\phi(x, y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Let $$x^2 = \theta_1,\ 2xy = \theta_2,\ y^2 = \theta_3,$$

$$\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + e\theta_3^2,$$

$$\omega = (x\xi + y\eta)^2 = x^2\theta_1 + xy\theta_2 + y^2\theta_3,$$

$$R = \theta_1\theta_3 - \frac{\theta_2^2}{4}.$$

Then Λ the discriminant of $\vartheta + 2\lambda R$ in respect to $\theta_1, \theta_2, \theta_3$

$$= \begin{vmatrix} a, & b, & c+\lambda \\ b, & c - \frac{1}{2}\lambda, & d \\ c+\lambda, & d, & e \end{vmatrix},$$

and I may remark that the relations between the several transformees of the invariantive plexuses formed by the minor determinant systems of Λ (in this, and in general for the case of an evenly-even index) may be found by treating $\vartheta + 2\lambda R$ as a quadratic function of the variables (in this case $\theta_1, \theta_2, \theta_3$) and applying the rule given by me in the *Philosophical Magazine* in my* paper "On the relation between the Minor Determinants of linearly-equivalent Quadratic Forms."† This second method, however, is not immediately applicable to the case of indices oddly even, that is of the form $4n+2$, to which the first method applies, equally as to the case $4n$; for if we make $2n+1 = p$ and $q = 2$, ω being of an odd degree, has no quadratic invariant; it has however a quadratic covariant, which will be of the second degree in respect to $\theta_1, \theta_2 \ldots \theta_{p+1}$ as well as in respect to ξ, η; and if we call this R and take the discriminant of $\vartheta + \lambda R$ in respect to the variables $\theta_1, \theta_2 \ldots \theta_{p+1}$, we shall obtain, as I am indebted to a remark of my valued friend M. Hermite for bringing under my notice, a very beautiful and interesting function of λ, of which all the coefficients will be contravariants of ϕ. Thus, let

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6,$$

[* p. 241 above.]

† Moreover, upon the supposition made in the text, the particular and absolute functions ϑ and ω may be treated in all respects as if they were functions characterizing quadratic loci, and any singularity in their relation will correspond to and denote a singularity in the given function ϕ to which ϑ refers. Thus, for instance, if ϕ be a function of x, y of the eighth degree, ϑ and ω will be quadratic functions of five letters each. Quadratic loci have no other singularity of relation than what corresponds to different species of contact. The number of contacts between loci, characterized by 5 letters, is 24 (see my paper‡ in the *Philosophical Magazine*, "On the contacts of lines and surfaces of the second order"). Consequently this mode of representing ϑ and ω will give rise to the discovery and specification of 24 different kinds of singularity in ϕ, and the analytical characteristics of each of them. But there of course may, and in fact will, exist other singularities in ϕ besides those which have their correspondencies in the relations of these quadratic concomitants. [‡ p. 237 above.]

make $\quad x^3 = \theta_1, \quad 3x^2y = \theta_2, \quad 3xy^2 = \theta_3, \quad y^3 = \theta_4,$

so that

$$\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + 2d\theta_1\theta_4 + g\theta_4^2 + 2f\theta_3\theta_4 + e\theta_3^2 + 2e\theta_4\theta_2,$$

$$\omega = (x\xi + y\eta)^3 = \theta_1\xi^3 + \theta_2\xi^2\eta + \theta_3\xi\eta^2 + \theta_4\eta^3,$$

$$R = \begin{vmatrix} 3\theta_1\xi + \theta_2\eta, & \theta_2\xi + \theta_3\eta \\ \theta_2\xi + \theta_3\eta, & \theta_3\xi + 3\theta_4\eta \end{vmatrix},$$

$$-R = \xi^2\theta_2^2 + \eta^2\theta_3^2 + \xi\eta\theta_2\theta_3 - 9\xi\eta\theta_1\theta_4 - 3\xi^2\theta_1\theta_3 - 3\eta^2\theta_2\theta_4.$$

Consequently the discriminant in respect to θ_1, θ_2, θ_3, θ_4 of $\vartheta - 2\lambda R$ becomes

$$\begin{vmatrix} a, & b, & c - 3\lambda\xi^2, & d - 9\lambda\xi\eta \\ b, & c + 2\lambda\xi^2, & d + \lambda\xi\eta, & e - 3\lambda\eta^2 \\ c - 3\lambda\xi^2, & d + \lambda\xi\eta, & e + 2\lambda\eta^2, & f \\ d - 9\lambda\xi\eta, & e - 3\lambda\eta^2, & f, & g \end{vmatrix}.$$

If this determinant be expanded as a function of λ, all the coefficients of the various powers of λ will be contravariants to the given function ϕ. The term involving λ^4 is zero. Let ξ become $-y$ and η become x, then the remaining terms (abstraction made of the powers of λ) become covariants of ϕ. The first term (the coefficient of λ^3) becomes ϕ itself; the last term is the catalecticant, and thus we see, in general, that for functions of x and y of an oddly-even degree, a whole series of covariants may be interpolated between the function and its catalecticant, the dimensions in respect of the coefficients of ϕ in arriving at each step increasing by 1 unit and the degree in respect of the variables diminishing by 2 units. This is consequently a much simpler and more available scale than one with which I have been long previously acquainted, and which applies alike to functions of any even degree.

Thus, let $\phi(x, y)$ be of $2k$ dimensions; form all the even emanants of ϕ, which will be all of the form $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{2\iota} \phi$, and take their respective catalecticants in respect to ξ and η. We shall in this way obtain a regular scale of covariants interpolated between the Hessian of ϕ (corresponding to $\iota = 1$) and the catalecticant of ϕ (corresponding to $\iota = k$). If ϕ be of the degree $2k + 1$, we shall have an analogous scale interpolated between the Hessian of ϕ and its canonizant; the latter term denoting the function which is the product of the $k + 1$ linear functions of x and y, the sum of whose $(2k + 1)$th powers is identically equal to ϕ*.

By means of the Theory of the Plexus we may obtain various representa-

* See Note (2) in Appendix. [p. 322 below.]

tions of the same invariant; thus, for example, if we take F a function of x, y of the fifth degree and form its Hessian H, that is

$$\begin{vmatrix} \dfrac{d^2F}{dx^2}, & \dfrac{d^2F}{dxdy} \\ \dfrac{d^2F}{dydx}, & \dfrac{d^2F}{dy^2} \end{vmatrix},$$

this will be a function of the sixth degree in x, y, and of the two orders in the coefficients. If we combine the two plexuses

$$\frac{dF}{dx}, \ \frac{dF}{dy}; \quad \frac{d^2H}{dx^2}, \ \frac{d^2H}{dxdy}, \ \frac{d^2H}{dy^2},$$

we shall have five equations between which x^4, x^3y, x^2y^2, xy^3, y^4 may be eliminated dialytically; the resultant will be of the $2+3.2$, that is the eighth order in the coefficients, and of the form $\square F - I_4^2$, where $\square F$ and I_4 are respectively the determinant and quintic invariant of F, each affected with a proper numerical multiplier (the "$B-A^2$" of my supplemental* essay on canonical forms) which, as Mr Cayley has remarked, may also be represented by the resultant of P; $\dfrac{dQ}{dx}$; $\dfrac{dQ}{dy}$ where P and Q are respectively the quadratic and cubic invariants in respect to ξ and η of $\left(\xi\dfrac{d}{dx}+\eta\dfrac{d}{dy}\right)^4 F$.

It will be well at this point to recapitulate in brief a method of elimination applicable to certain systems of functions published by me many years since in the *Philosophical Magazine*, and to compare this method with that afforded by the theory of the plexus for finding an invariant for each of the very same systems, possessing all the external characters, formed in a precisely similar manner to, and not impossibly identical *with*, the resultant of every such system. I shall devote my first moments of leisure to the ascertainment of this last most important point, as to the identity or otherwise of the plexus-invariant with the resultant. Take the case of three functions of x, y, z (say ϕ, ψ, ω) each of the same degree n; to fix the ideas, suppose $n=3$: there are two purely algebraical processes (modifications of the same method and leading to identical results) by which the resultant of ϕ, ψ, ω may be found. I shall call these processes the first and second respectively.

First process: Write

$$\begin{aligned}\phi &= x^2P + yQ + zR,\\ \psi &= x^2P' + yQ' + zR',\\ \omega &= x^2P'' + yQ'' + zR'',\end{aligned}$$

decompositions which may be effected in an infinite variety of manners, so that P, Q, R shall be integer functions of x, y, z; take the linear resultant of ϕ, ψ, ω, in respect to x^2, y, z, which call $H_{2,1,1}$; this will evidently be

[* p. 205 above.]

of $9-4$, that is, of 5 dimensions. Form analogously the functions $H_{1,2,1}$, $H_{1,1,2}$; $H_{2,1,1}$, $H_{1,2,1}$, $H_{1,1,2}$ constitute an auxiliary system of functions which vanish when ϕ, ψ, ω vanish together; combine this auxiliary system with the augmentative system

$$\begin{array}{cccccc} x^2\phi, & y^2\phi, & z^2\phi, & xy\phi, & yz\phi, & zx\phi, \\ x^2\omega, & y^2\omega, & z^2\omega, & xy\omega, & yz\omega, & zx\omega, \\ x^2\psi, & y^2\psi, & z^2\psi, & xy\psi, & yz\psi, & zx\psi. \end{array}$$

We shall thus have in all $3+3\times 6$, that is, 21 functions into which the 21 terms x^5, x^4y, x^4z, &c. enter linearly: the linear resultant of these 21 functions is the resultant of ϕ, ψ, ω, clear of all extraneousness.

Second process: Write

$$\phi = x^3P + yQ + zR,$$
$$\psi = x^3P' + yQ' + zR',$$
$$\omega = x^3P'' + yQ'' + zR'',$$

and, as before, take the linear resultant $H_{3,1,1}$, which will however be of $9-5$, that is, of only 4 dimensions.

Again, take

$$\phi = x^2L + y^2M + zN,$$
$$\psi = x^2L' + y^2M' + zN',$$
$$\omega = x^2L'' + y^2M'' + zN'',$$

and form the determinant $H_{2,2,1}$; we shall thus have the auxiliary system

$$H_{3,1,1},\quad H_{1,3,1},\quad H_{1,1,3},\quad H_{2,2,1},\quad H_{2,1,2},\quad H_{1,2,2}.$$

Let this be combined with the augmentative system

$$x\omega,\ y\omega,\ z\omega;\quad x\phi,\ y\phi,\ z\phi;\quad x\psi,\ y\psi,\ z\psi.$$

Between these $6+9$, that is, 15 functions, the 15 terms x^4, x^3y, x^3z, &c. may be linearly eliminated, and the resultant thus obtained will be precisely the same as that got by the preceding process.

Here we have 6 auxiliaries and 6 augmentatives; the auxiliaries are of three dimensions in respect to the coefficients of ϕ, ψ, ω; the augmentatives of one dimension only; in the former process there were 3 auxiliaries and 18 augmentatives, $6\times 3+9=27=3\times 3+18$.

Now let this method be compared with the following:

First process: Take the 18 augmentatives $x^2\phi$, $x^2\omega$, $x^2\psi$, &c. as in the first process of the algebraical method above explained; but in place of the 3 auxiliaries therein given, take another system of 9 as follows:

Write the determinant

$$\begin{vmatrix} \frac{d\phi}{dx}, & \frac{d\phi}{dy}, & \frac{d\phi}{dz} \\ \frac{d\psi}{dx}, & \frac{d\psi}{dy}, & \frac{d\psi}{dz} \\ \frac{d\omega}{dx}, & \frac{d\omega}{dy}, & \frac{d\omega}{dz} \end{vmatrix} = R;$$

$\frac{dR}{dx}$, $\frac{dR}{dy}$, $\frac{dR}{dz}$ form a concomitantive plexus; the 18 augmentatives form another; the linear resultant of these two plexuses will be an invariant of ϕ, ψ, ω, and of precisely the same dimensions as the resultant last found; if they are not identical it will be indeed a matter of exceeding wonder, and even more interesting than if they should be proved so to be.

Second process: Combine the augmentative plexus

$$x\omega,\ y\omega,\ z\omega;\quad x\phi,\ y\phi,\ z\phi;\quad x\psi,\ y\psi,\ z\psi,$$

with the differential plexus

$$\frac{d^2R}{dx^2},\ \frac{d^2R}{dxdy},\ \frac{d^2R}{dy^2},\ \frac{d^2R}{dydz},\ \frac{d^2R}{dz^2},\ \frac{d^2R}{dzdx},$$

we thus obtain a linear resultant in a manner precisely similar to that afforded by the second process of our algebraical method.

In general, if ϕ, ψ, ω be of the degrees n, n, n, as there are two algebraical varieties of the linear method for finding the resultant, so are there two varieties of the concomitantive method for finding the resembling invariant. In both methods the augmentatives are identical; the only difference being in the auxiliary system.

In the first process the augmentative system will be got by operating upon each of the functions ϕ, ψ, ω, with the multipliers x^{n-1}, y^{n-1}, z^{n-1}, and the other homogeneous products of x, y, z; the auxiliary system by operating upon R with the symbolical multipliers $\left(\frac{d}{dx}\right)^{n-2}$, $\left(\frac{d}{dy}\right)^{n-2}$, $\left(\frac{d}{dz}\right)^{n-2}$, and the other homogeneous products of $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ of the degree $n-2$.

In the second process the augmentative system is formed by the aid of the multipliers x^{n-2}, y^{n-2}, z^{n-2}, &c., and the auxiliary system by aid of

$$\left(\frac{d}{dx}\right)^{n-1},\ \left(\frac{d}{dy}\right)^{n-1},\ \left(\frac{d}{dz}\right)^{n-1},\ \text{\&c.}$$

For the particular case of $n=2$ the first process of the concomitantive method is merely an application under its most symmetrical form of the first process of the general algebraical method. The second process of the concomitantive method for this same case (at least when ϕ, ψ, ω are the partial differential coefficients of the same function of the third degree) has been shown by Dr Hesse to give the resultant, so that for this case, at all events, we know that each concomitantive auxiliary must be a linear function of the augmentatives and the algebraical auxiliaries.

Again, if we go to the system where ϕ, ψ, ω are of the respective degrees n, n, $n+1$. In the algebraical method (for applying which there are no longer two, but one only process), the augmentative system is obtained by multiplying ϕ by the homogeneous products of x^{n-1}, $x^{n-1}y$, $x^{n-1}z$, &c., ψ by the like products, and ω by the homogeneous products x^{n-2}, $x^{n-2}y$, &c. The auxiliary system is made up of functions of the general form

$$H_{p,q,r} \text{ where } p+q+r=n+2,$$

$H_{p,q,r}$ being the determinant obtained by writing

$$\phi = Lx^p + My^q + Nz^r,$$
$$\psi = L'x^p + M'y^q + N'z^r,$$
$$\omega = L''x^p + M''y^q + N''z^r.$$

And in like manner for the case of ϕ, ψ, ω, being of the respective degrees n, n, $n-1$, the augmentative system is obtained by affecting ϕ, ψ each with multipliers x^{n-2}, $x^{n-2}y$, &c., and ω with the multipliers x^{n-1}, $x^{n-1}y$, &c.

The number of functions (for either case) in the augmentative and auxiliary plexuses thus obtained will be found to be exactly equal to the number of terms in each such function, as shown by me in the paper alluded to. Let this be compared with the transcendental method (I use this word at this point in preference to concomitantive, because in fact the algebraical and differential auxiliary systems are both alike concomitantive plexuses to ϕ). For the case of n, n, $n+1$, the Jacobian determinant R of ϕ, ψ, ω will be of the degree $3n-2$, and the system $\left(\frac{d}{dx}\right)^{n-1} R$, $\left(\frac{d}{dx}\right)^{n-2} \left(\frac{d}{dy}\right) R$, &c. combined with the augmentative systems

$$x^{n-2}\omega, \quad x^{n-3}y\omega, \quad \&c.$$
$$x^{n-1}\phi, \quad x^{n-2}y\phi, \quad \&c.$$
$$x^{n-1}\psi, \quad x^{n-2}y\psi, \quad \&c.$$

will give an invariant resembling (at least in generation and form) if not identical with the resultant of ϕ, ψ, ω. For the case of ϕ, ψ, ω being of the degrees n, n, $n-1$, the Jacobian R is of the degree $3n-4$ and

$$\left(\frac{d}{dx}\right)^{n-2} R, \quad \left(\frac{d}{dx}\right)^{n-3} \frac{d}{dy} R, \ \&c.$$

is the system which, combined with the augmentative systems

$$x^{n-2}\phi, \quad x^{n-3}y\phi, \text{ \&c.}$$
$$x^{n-2}\psi, \quad x^{n-3}y\psi, \text{ \&c.}$$
$$x^{n-1}\omega, \quad x^{n-2}y\omega, \text{ \&c.}$$

will produce the resembling invariant.

Finally, for the last and more special case which the algebraical method applies to, namely of ϕ, ψ, ω, θ, four quadratic functions of x, y, z, t, there can be here little doubt (upon the first impression) that in place of the algebraically obtained plexus

$$H_{2,1,1,1}, \quad H_{1,2,1,1}, \quad H_{1,1,2,1}, \quad H_{1,1,1,2},$$

may be substituted the differential plexus

$$\frac{dR}{dx}, \quad \frac{dR}{dy}, \quad \frac{dR}{dz}, \quad \frac{dR}{dt},$$

which, combined with the augmentatives

$$x\phi,\ x\psi,\ x\omega,\ x\theta; \quad y\phi,\ y\psi,\ y\omega,\ y\theta; \quad z\phi,\ z\psi,\ z\omega,\ z\theta; \quad t\phi,\ t\psi,\ t\omega,\ t\theta,$$

will render possible the dialytic elimination of the 20 homogeneous products

$$x^3, \quad x^2y, \quad x^2z, \quad x^2t, \quad xyz, \quad y^3, \text{ \&c. \&c.*}$$

Upon precisely the same principles may be verified instantaneously the method given by Hesse (without demonstration) for finding the polar reciprocal of lines of the third and fourth orders, at least to the extent of seeing that the functions obtained by his methods are contravariants (of the right degree and order) of the function from which they are derived. The polar reciprocal to a *surface* of the third degree may be obtained in the same manner.

Let $\phi(x, y, z, t)$ be the characteristic of such a surface. If we form a differential plexus of the first emanant of ϕ taken together with the concomitant $w = x\xi + y\eta + z\zeta + t\theta$, by operating with

$$\frac{d}{dx}, \ \frac{d}{dy}, \ \frac{d}{dz}, \ \frac{d}{dt} \text{ upon } \left(\xi'\frac{d}{dx} + \eta'\frac{d}{dy} + \zeta'\frac{d}{dz} + \theta'\frac{d}{dt}\right)(\phi + \lambda w),$$

and combining this plexus with $x\xi' + y\eta' + z\zeta' + t\theta'$, the resultant taken in respect to ξ', η', ζ', θ' (say R) will (according to the law of synthesis) be a

* Subsequent reflection induces me to reject as very improbable the (at first view likely) conjecture of the identity of the resultant with the invariant which simulates its form, except in the proved cases of three quadratic functions and the strongly resembling case of four quadratic functions last adverted to in the text above. Did this identity obtain, analogy would indicate that the catalecticant of the Hessian of two homogeneous functions of the same degree in x, y should be identical with their resultant, which is easily demonstrated to be false, except when the functions are of the third degree.

contravariant to the system $\phi + \lambda w$ and w, and therefore to ϕ, because w is itself a concomitant to ϕ. R is of the third degree in x, y, z, t, as also in the coefficients of ϕ. If we form a differential plexus of $R + \mu w$ analogous to that formed above with $\phi + \lambda w$, and combine these two plexuses with the augmentative system xw, yw, zw, tw, there will be $4+4+4$, that is, 12 functions containing the 12 terms x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt, λ, μ, and the dialytic resultant, which will be found to be a contravariant of the twelfth degree in ξ, η, ζ, θ, and of the twelfth order in respect of the coefficients of ϕ, will be (there can be little doubt) the polar reciprocal to the characteristic ϕ.

A few remarks upon the analytical character of a polar reciprocal may be not out of place here. If ϕ be any homogeneous function of the degree m of any number (n) of variables ($x, y \ldots z$), the object of the theory of polar reciprocals is to discover what is the relation between $\xi, \eta \ldots \zeta$ expressed in the simplest terms such that, when this equation is satisfied, $\xi x + \eta y + \ldots + \zeta z = 0$ will be tangential to $\phi = 0$. In order for this to take effect it is necessary that when any one of the variables z is expressed in terms of the others $\ldots y, x$, and this value established in ϕ, the discriminant of ϕ, so transformed, should be zero. Consequently the characteristic of the polar reciprocal to ϕ is that rational integral function which is common to all the discriminants obtained by expressing ϕ (by aid of the equation $\xi x + \eta y + \ldots + \zeta z$) as a function of any $(n-1)$ of the variables. Let I_x be any invariant whatever of the order r of ϕ_x (meaning by this last symbol what ϕ becomes when x is eliminated), and $I_y \ldots I_z$ the corresponding invariants when $y \ldots z$ respectively are eliminated; I_x will evidently be of the form $\dfrac{E_x}{(\xi)^{mr}}$, the numerator being an integer of r dimensions in the coefficients of ϕ and of mr dimensions in respect of $\xi, \eta \ldots \zeta$; and by the fundamental definition of invariants it may easily be shown that

$$I_x : I_y : \ldots : I_z :: \frac{1}{\xi^{\frac{mr}{n-1}}} : \frac{1}{\eta^{\frac{mr}{n-1}}} : \ldots : \frac{1}{\zeta^{\frac{mr}{n-1}}} *,$$

and therefore

$$\frac{E_x}{\xi^p} = \frac{E_y}{\eta^p} = \ldots = \frac{E_z}{\zeta^p}, \quad \text{where } p = \frac{m(n-2)r}{n-1}.$$

Consequently all these quotients must be essentially integer, and any one of them will be of the order r in respect of the coefficients of ϕ and of the

* We see indirectly from this, that for a function of $(n-1)$, say γ, variables of the degree m, an invariant of the order r must be subject to the condition that $\dfrac{mr}{\gamma}$ = an integer. This is easily shown upon independent grounds; when $\gamma = 2$, $\dfrac{mr}{\gamma}$ must be not merely an integer but an *even integer*, and doubtless some analogous law applies to the general case.

degree $\frac{mr}{n-1}$ in respect of $\xi, \eta \dots \zeta$. Consequently the polar characteristic of ϕ, which is the common factor of the *discriminants* of $I_x, I_y \dots I_z$ (for which species of invariant r evidently is equal to $(n-1)(m-1)^{n-2}$, the function being in fact the discriminant of a function of the mth degree of $(n-1)$ variables), will be of the order $(n-1)(m-1)^{n-2}$ in respect of the coefficients of ϕ and of the degree $m(m-1)^{n-2}$ in respect of the contragredients $\xi, \eta \dots \zeta$.

As to what relates to the reciprocity which exists between ϕ and its polar reciprocal ψ, this is included in a much higher theory of elimination, one proposition of which may be enunciated somewhat to the effect following, namely that if ϕ be a homogeneous function of $x, y \dots z$, and ω of $x, y \dots z, u, v \dots w$, and if, by aid of the equations

$$\phi = 0,$$

$$\frac{d\phi}{dx} + \lambda \frac{d\omega}{dx} = 0,$$

$$\frac{d\phi}{dy} + \lambda \frac{d\omega}{dy} = 0,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{d\phi}{dz} + \lambda \frac{d\omega}{dz} = 0,$$

$x, y \dots z$ be eliminated and the resultant be called ψ, then the effect of performing a similar operation upon ψ, ω, with respect to $u, v \dots w$, as that just above indicated for the system ϕ, ω, with respect to $x, y \dots z$, will be to give a resultant, one factor of which will be the primitive function ϕ over again. There is some reason for supposing that polar reciprocals, which are scarcely ever (if ever, except indeed for quadratic functions) the simplest contravariants to a given function, may be expressed algebraically by means of the simpler contravariants, in the same way as discriminants admit (in many, if not in all cases, with the same exception as above) of being represented as algebraical functions of invariants of a lower order or simpler form.

I close this section with the remark that every complete and unambiguous system of functions of the constants in a given form or set of forms *characteristic** of any singularity absolute or relative in such form or forms must

* I repeat here that a function or system of functions which severally equated to zero express unequivocally and completely the existence of any position or negation, is termed the characteristic of such position or negation. Thus for example the resultant of a group of equations is the characteristic of the possibility of their coexistence. The discriminant of a function of two variables is the characteristic of its possession of two equal factors; the catalecticant is the characteristic of its decomposability into the sum of a defined number of powers of linear functions of the variables, &c.

constitute an invariantive plexus or set of invariantive plexuses. The system unambiguously characteristic of a singularity of an order n will (except when $n=1$) almost universally consist of far more than n functions, subject of course to the existence of syzygetic* relations between any $(n+1)$ of such functions. The existence of multiple roots of a function of two variables is a specific, but by no means a peculiar case of singularity, and requires, for its complete and systematic elucidation, to be treated in connexion with the general theory of the subject.

SECTION III. *On Commutants.*

The simplest species of commutant is the well-known common determinant.

If we combine each of the n letters $a, b \ldots l$ with each of the other n, $\alpha, \beta \ldots \lambda$, we obtain n^2 combinations which may be used to denote the terms of a determinant of n lines and columns, as thus:

$$\begin{array}{l} a\alpha,\quad a\beta \ldots a\lambda, \\ b\alpha,\quad b\beta \ldots b\lambda, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ l\alpha,\quad l\beta \ldots l\lambda. \end{array}$$

It must be well understood that the single letters of either set are mere umbræ, or shadows of quantities, and only acquire a real signification when one letter of one set is combined with one of the other set. Instead of the inconvenient form above written, we may denote the determinant more simply by the matrix

$$\begin{array}{l} a,\quad b,\quad c \ldots l, \\ \alpha,\quad \beta,\quad \gamma \ldots \lambda; \end{array}$$

and to find the expanded value of such a matrix the rule is evidently to take one of the lines in all its 1, 2, 3 ... n different forms, arising from the permutations of the letters (or umbræ) which it contains; and then form the product of the n quantities formed by the combination of the respective pairs of letters in the same vertical column, affecting such product with the sign of + or − according to the rule, that all products corresponding to arrangements of the terms subject to the permutation derivable from one another by an even number of interchanges are of the same, and by an odd number of interchanges of a contrary sign. If both lines are permuted and a similar rule applied, with the additional circumstance that the sign of the products

* Rational integer functions which admit of being multiplied severally by other rational integer functions such that the sum of the products is identically zero, are said to be "syzygetically related."

is made to depend on the product of the algebraical signs due to the respective arrangements in the two lines of umbræ, it is evident that the result will be the same as when only one line is put into motion, save and except that a numerical factor $1 . 2 . 3 \ldots n$ will affect each term. If the two sets of umbræ $a, b, c \ldots l$; $\alpha, \beta, \gamma \ldots \lambda$ be taken identical, and if it be convened that the order of the combination of any two letters shall not affect the value of the quantity thereby denoted, $\begin{matrix} a, b, c \ldots l \\ a, b, c \ldots l \end{matrix}$ will denote a symmetrical determinant.

If instead of two lines of umbræ, three or more be taken, the same principle of solution will continue to be applicable. Thus, if there be a matrix of any even number r of lines each of n umbræ,

$$\begin{matrix} a_1, & b_1 \ldots l_1, \\ a_2, & b_2 \ldots l_2, \\ \ldots & \ldots\ldots \\ a_r, & b_r \ldots l_r, \end{matrix}$$

the first may be supposed to remain stationary, and the remaining $(r-1)$ lines each be taken in $1, 2 \ldots n$ different orders; every order in each line will be accompanied by its appropriate sign $+$ or $-$; and each different grouping in each line will give rise to a particular grouping of the letters read off in columns. The value of the commutant expressed by the above matrix will therefore consist of the sum of $(1 . 2 \ldots n)^{r-1}$ terms, each term being the product of n quantities respectively symbolized by a group of r letters and affected with the sign $+$ or $-$ according as the number of negative signs in the total of the arrangements of the lines (from the columnar reading off of which each such term is derived) is even or odd.

For example, the value of

$$\begin{matrix} a, & b, \\ c, & d, \\ e, & f, \\ g, & h, \end{matrix}$$

will be found by taking the $(1 . 2)^3$ arrangements, as below,

$$\begin{matrix} a, b, & a, b, & a, b, & a, b, & a, b, & a, b, & a, b, & a, b, \\ c, d, & d, c, & c, d, & d, c, & c, d, & d, c, & c, d, & d, c, \\ e, f, & e, f, & f, e, & f, e, & e, f, & e, f, & f, e, & f, e, \\ g, h, & g, h, & g, h, & g, h, & h, g, & h, g, & h, g, & h, g. \end{matrix}$$

The signs of c, d; e, f; g, h being supposed $+$, those of d, c; f, e and h, g will be each $-$. Consequently the sum of the terms will be expressed by

$$aceg \times bdfh - adeg \times bcfh - acfg \times bdeh + adfg \times bceh$$
$$- aceh \times bdfg + adeh \times bcfg + acfh \times bdeg - adfh \times bceg.$$

Commutants thus formed may be termed total commutants, because the entire of each line is made to pass through all its possible forms of arrangement. In total commutants it is necessary that the number of lines r be even; for if taken odd, on making all the r lines to change, instead of obtaining $1 . 2 \dots n$ lines, the result obtained when all but one are made to change, it will be found that the latter will be repeated $\frac{1}{2}(1 . 2 \dots n)$ times with the sign $+$, and $\frac{1}{2}(1 . 2 \dots n)$ times with the sign $-$, so that the algebraical sum of the terms will be zero. Moreover the commutants of the species above described, besides being total, are simple, inasmuch as all the umbræ to be termed consist of single letters.

My first proposition in the application of the theory of commutants to that of forms is as follows:

Let ϕ be a function homogeneous and linear in respect to an even number r of any systems whatever of variables, as

$$x_1,\ y_1 \dots t_1;\quad x_2,\ y_2 \dots t_2;\quad x_r,\ y_r \dots t_r.$$

Form the commutant

$$\begin{matrix} \dfrac{d}{dx_1}, & \dfrac{d}{dy_1} \cdots & \dfrac{d}{dt_1}, \\ \dfrac{d}{dx_2}, & \dfrac{d}{dy_2} \cdots & \dfrac{d}{dt_2}, \\ \dots & \dots & \dots \\ \dfrac{d}{dx_r}, & \dfrac{d}{dy_r} \cdots & \dfrac{d}{dt_r}. \end{matrix}$$

Let the general term of this commutant, expanded, be called

$$F_{\theta_1} \times F_{\theta_2} \times \dots \times F_{\theta_r},$$

then is

$$\Sigma F_{\theta_1} . \phi \times F_{\theta_2} . \phi \times \dots \times F_{\theta_r} . \phi$$

a covariant or invariant*, as the case may be, of ϕ.

Be it observed that the march of the substitution for the different sets of variables in the above proposition is supposed to be perfectly independent. All the systems but one may undergo linear transformation, or they may all undergo distinct and disconnected transformations at the same time, and the proposition still continue applicable. It will however evidently be no less applicable should the march of substitution for any of the systems become cogredient or contragredient to that of any other systems.

If we suppose ϕ to be a function of an even degree r of a single system of n variables $x, y \dots t$, so that the r systems x_1, y_1, &c., x_2, y_2, &c. ... x_r, y_r, &c. become identical, we can at once infer from the above scheme the existence and mode of forming an invariant to ϕ of the order n. This last appears

[* See below, p. 324.]

for the case $n = 2$, and ought, for all other values of n, to have been known* to the author of the immortal discovery of invariants, termed by him hyperdeterminants, in the sense which, according to the nomenclature here adopted, would be conveyed by the term hyperdiscriminants.

Before proceeding to discuss the theory of compound total commutants, or enlarging upon that of partial commutants, I shall make an interesting application of the preceding general proposition to the discovery of Aronhold's S and T, the two invariants respectively of the fourth and sixth orders appertaining to a homogeneous cubic function (say F) of three variables x, y, z. These may be termed respectively H_4 and H_6. As to H_6 a theoretically possible but eminently prolix and ungraceful method immediately presents itself, namely to take $F^2 = G$, and after forming the commutant with six lines,

$$\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \end{matrix}$$

to operate with the 6^5 ternary products of which this is made up upon G: the result being an invariant of G, will be so to F, and being of the third degree in respect to the coefficients of G, will be of the sixth in respect to those of F. It will evidently therefore be H_6, or at least a numerical multiple of H_6, the form of which, inasmuch as the only other invariant is H_4, we know in form to be unique. But the general theorem affords another and probably the

* That this was not known explicitly to and should have escaped the penetration of the sagacious author of the theory, and those who had studied his papers, must be attributed to the imperfection of the notation heretofore employed for denoting the coefficients of a homogeneous polynomial function. The umbral method of denoting such a function ϕ of the degree r under the form of $(ax+by+\ldots+cz)^r$, which is equivalent to, but a more compendious and independent mode of mentally conceiving and handling the representation

$$\left(x\frac{d}{dx}+y\frac{d}{dy}+\ldots\ldots+z\frac{d}{dz}\right)\phi,$$

exhibits the true internal constitution of such functions, and necessarily leads to the discovery of their essential properties and attributes.

most practically compendious* solution as regards H_6, of which the question admits.

Let G† represent the mixed concomitant to F formed by the bordered determinant

$$\begin{vmatrix} \frac{d^2F}{dx^2}, & \frac{d^2F}{dxdy}, & \frac{d^2F}{dxdz}, & \xi \\ \frac{d^2F}{dydx}, & \frac{d^2F}{dy^2}, & \frac{d^2F}{dydz}, & \eta \\ \frac{d^2F}{dzdx}, & \frac{d^2F}{dzdy}, & \frac{d^2F}{dz^2}, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}.$$

G is a function of the second order as to x, y, z, and of the like order in respect to ξ, η, ζ, which two systems will be respectively cogredient and contragredient in respect to the x, y, z system in F. In other words, which is all we need to look to, G is a concomitant of F, and so also will be

$$G + \lambda (x\xi + y\eta + z\zeta)^2,$$

which may be termed H. Form now the commutant

$$\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, \end{matrix}$$

this being applied to H will give an invariant (the fact that the march of the substitutions for the systems x, y, z; ξ, η, ζ is contrary, being completely immaterial to the applicability of the general theorem above given);

* Having since this was printed been favoured with a view of some of the proof-sheets of Mr Salmon's most valuable Second Part of his *System of Analytical Geometry* (about to appear, and which is calculated, in my opinion, to awaken a higher idea of and excite a new taste for geometrical researches in this country), I find that I am mistaken in this point; the less symmetrical method operated with by Mr Salmon being decidedly the shortest for practically obtaining S and T in the general case. Symmetry, like the grace of an eastern robe, has not unfrequently to be purchased at the expense of some sacrifice of freedom and rapidity of action.

† G is the mixed concomitant to the given cubic function, which is halfway (so to speak) between it and its polar reciprocal. In fact, when the operation is repeated upon G, which was executed upon the given function to obtain G (that is, when we *border* the Hessian of G in respect to x, y, z, vertically and horizontally with the column and line ξ, η, ζ) the determinant thereby represented becomes the polar reciprocal to the given function.

the commutant so formed will be a cubic function of λ, in which the coefficient of λ^3 is a numerical quantity, that of λ^2 is zero, that of λ is H_4 and the constant term is H_6.

Thus for example let $F = x^3 + y^3 + z^3 + 6mxyz$, then

$$G = \begin{vmatrix} x, & mz, & my, & \xi \\ mz, & y, & mx, & \eta \\ my, & mx, & z, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}$$

and therefore

$$H = \Sigma \{(\lambda - m^2)\, x^2\xi^2 + (\lambda + m^2)\, 2yz\eta\zeta + yz\xi^2 - 2mx^2\eta\zeta\},$$

the Σ implying the sum of similar terms with reference to the interchanges between x, ξ; y, η; z, ζ.

In developing the commutant above, the first line may be kept in a fixed position; for the sake of brevity, (x), (y), (z); (ξ), (η), (ζ) may be written in the place of

$$\frac{d}{dx}, \ \frac{d}{dy}, \ \frac{d}{dz}; \ \frac{d}{d\xi}, \ \frac{d}{d\eta}, \ \frac{d}{d\zeta},$$

and it will readily be seen that the only effective arrangements will be as underwritten:

$$\begin{array}{lll}
(x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) \\
(x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) \\
(\xi)\,(\eta)\,(\zeta) & (\eta)\,(\zeta)\,(\xi) & (\zeta)\,(\xi)\,(\eta) \\
(\xi)\,(\eta)\,(\zeta) & (\zeta)\,(\xi)\,(\eta) & (\eta)\,(\zeta)\,(\xi)
\end{array}$$

$$\begin{array}{llllll}
(x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) \\
(x)\,(z)\,(y) & (x)\,(z)\,(y) & (\zeta)\,(\eta)\,(\xi) & (z)\,(y)\,(x) & (y)\,(x)\,(z) & (y)\,(x)\,(z) \\
(\xi)\,(\eta)\,(\zeta) & (\xi)\,(\zeta)\,(\eta) & (\xi)\,(\eta)\,(\zeta) & (\zeta)\,(\eta)\,(\xi) & (\xi)\,(\eta)\,(\zeta) & (\eta)\,(\xi)\,(\zeta) \\
(\xi)\,(\zeta)\,(\eta) & (\xi)\,(\eta)\,(\zeta) & (\zeta)\,(\eta)\,(\xi) & (\xi)\,(\eta)\,(\zeta) & (\eta)\,(\xi)\,(\zeta) & (\xi)\,(\eta)\,(\zeta)
\end{array}$$

$$\begin{array}{llllll}
(x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) \\
(x)\,(z)\,(y) & (x)\,(z)\,(y) & (z)\,(x)\,(y) & (z)\,(y)\,(x) & (y)\,(x)\,(z) & (y)\,(x)\,(z) \\
(\eta)\,(\zeta)\,(\xi) & (\zeta)\,(\eta)\,(\xi) & (\xi)\,(\zeta)\,(\eta) & (\zeta)\,(\xi)\,(\eta) & (\eta)\,(\zeta)\,(\xi) & (\xi)\,(\zeta)\,(\eta) \\
(\zeta)\,(\eta)\,(\xi) & (\eta)\,(\zeta)\,(\xi) & (\zeta)\,(\xi)\,(\eta) & (\xi)\,(\zeta)\,(\eta) & (\xi)\,(\zeta)\,(\eta) & (\eta)\,(\zeta)\,(\xi)
\end{array}$$

$$\begin{array}{llllll}
(x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) & (x)\,(y)\,(z) \\
(y)\,(z)\,(x) & (z)\,(x)\,(y) & (y)\,(z)\,(x) & (y)\,(z)\,(x) & (z)\,(x)\,(y) & (z)\,(x)\,(y) \\
(\zeta)\,(\xi)\,(\eta) & (\eta)\,(\zeta)\,(\xi) & (\xi)\,(\eta)\,(\zeta) & (\eta)\,(\zeta)\,(\xi) & (\xi)\,(\eta)\,(\zeta) & (\zeta)\,(\xi)\,(\eta) \\
(\xi)\,(\zeta)\,(\eta) & (\eta)\,(\zeta)\,(\xi) & (\eta)\,(\zeta)\,(\xi) & (\xi)\,(\eta)\,(\zeta) & (\zeta)\,(\xi)\,(\eta) & (\xi)\,(\eta)\,(\zeta)
\end{array}$$

The signs of the four lines in each of these arrangements are two alike, and two contrary to the signs of the correspondent lines in the first arrangement; hence the effective sign is the same for all, and the result, after rejecting from each term the common factor -16, is seen, from inspection, to be

$$4(\lambda - m^2)^3 - 8m^3 + 6(\lambda - m^2)(\lambda + m^2)^2 - 12m(\lambda + m^2) + 2(\lambda + m^2)^3 + 1,$$

which is equal to

$$12\lambda^3 + 0\,.\,\lambda^2 - 12(m - m^4)\lambda + 1 - 20m^3 - 8m^6;$$

here the coefficients $m - m^4$ and $1 - 20m^3 - 8m^6$ are the two invariants (Aronhold's S and T) for the canonical form operated upon; and it will be observed that

$$(1 - 20m^3 - 8m^6)^2 + 64(m - m^4)^3 = (1 + 8m^3)^3,$$

which is easily proved to be the discriminant of

$$x^3 + y^3 + z^3 + 6mxyz.$$

It may however be observed, that this is not the discriminant of the function in λ just found, as reasons of analogy* might have suggested it probably would be: in order that this might be the case, the coefficient of λ^3 should be 4 instead of 12, and of λ, $m - m^4$ instead of $m^4 - m$. There is ground for supposing that another function of λ may be found by a different method, in which this relation will take effect.

The theorem above given for simple total commutants admits of an interesting application to the general case of a function F of the nth degree, in respect to each of two independent systems of two variables x, y; ξ, η. Let F be symbolically represented by $(ax + by)^n(\alpha\xi + \beta\eta)^n$, so that $a^n\alpha^n$ represents the coefficient of $x^n\xi^n$, $na^{n-1}b\alpha^n$ of $x^{n-1}y\xi^2$, &c. &c.; then the commutant

$$\begin{array}{lr} a,\ b, & (1) \\ a,\ b, & (2) \\ \cdots & \\ a,\ b, & (n) \\ \alpha,\ \beta, & (1) \\ \alpha,\ \beta, & (2) \\ \cdots & \\ \alpha,\ \beta, & (n) \end{array}$$

will represent a quadratic invariant of F, which will contain $(n+1)^2$ coefficients. By expanding this commutant we obtain a general expression for the invariant under a very interesting form.

* The biquadratic function of x, y having only one parameter, and therefore two invariants, its theory possesses striking analogies to the theory of the cubic function of three letters. The function in λ which gives these invariants for the first-named function, according to the method given in the first section, has the same discriminant as the function itself.

I now proceed to give the general theorem for compound total commutants as applicable to the discovery of invariants.

Let there be a function of m disconnected classes of systems of variables; let the systems in the same class be supposed all distinct but cogredient with one another. The function is supposed to be linear in respect to each system in each class, and the number of systems is the same for all the classes, and the number of variables the same in each system. This function may then be represented symbolically under the form

$$({}^1a_1 . {}^1x_1 + {}^1b_1 . {}^1y_1 + \ldots + {}^1l_1 . {}^1t_1)({}^1a_2 . {}^1x_2 + {}^1b_2 . {}^1y_2 + \ldots + {}^1l_2 . {}^1t_2)$$
$$\ldots ({}^1a_n . {}^1x_n + {}^1b_n . {}^1y_n + \ldots {}^1l_n . {}^1t_n)$$
$$\times ({}^2a_1 . {}^2x_1 + {}^2b_1 . {}^2y_1 + \ldots + {}^2l_1 . {}^2t_1)({}^2a_2 . {}^2x_2 + {}^2b_2 . {}^2y_2 + \ldots + {}^2l_2 . {}^2t_2)$$
$$\ldots ({}^2a_n . {}^2x_n + {}^2b_n . {}^2y_n + \ldots {}^2l_n . {}^2t_n)$$
$$\times \text{\&c.}$$
$$\times ({}^pa_1 . {}^px_1 + {}^pb_1 . {}^py_1 + \ldots + {}^pl_1 . {}^pt_1)({}^pa_2 . {}^px_2 + {}^pb_2 . {}^py_2 + \ldots + {}^pl_2 . {}^pt_2)$$
$$\ldots ({}^pa_n . {}^px_n + {}^pb_n . {}^py_n + \ldots {}^pl_n . {}^pt_n).$$

In this expression the $x, y \ldots t$'s are all real, but the $a, b \ldots l$'s all umbral; in fact, ${}^fa_g$, ${}^fb_g$, &c. may be understood to denote $\frac{d}{d^fx_g}$, $\frac{d}{d^fy_g}$, &c.

The n systems of variables in each of the sets above written are supposed to be cogredient *inter se.*

Take the symbolical product of the first set, first making *for the moment*

$${}^1x_1 = {}^1x_2 = \ldots {}^1x_n = x, \text{ \&c. \&c.,} \quad {}^1t_1 = {}^1t_2 = \ldots {}^1t_n = t;$$

and let the coefficients of the several terms

$$x^n,\ x^{n-1}y \ldots \text{\&c.,}$$

be called
$${}^1A_1,\ {}^1A_2 \ldots {}^1A_\mu,$$

where μ is the number of terms contained in a homogeneous function of the nth degree of the m variables $x, y \ldots t$. In like manner proceed with each of the lines, and then write down the commutant

$$\begin{matrix} {}^1A_1, & {}^1A_2 \ldots {}^1A_\mu, \\ {}^2A_1, & {}^2A_2 \ldots {}^2A_\mu, \\ \ldots\ldots & \ldots\ldots\ldots \\ {}^pA_1, & {}^pA_2 \ldots {}^pA_\mu. \end{matrix}$$

This commutant is an invariant of F: it will of course be remembered that, unless p is even, the commutant vanishes.

Thus, for example, take two sets of two systems of two variables: in all four systems,

$$x, y; \quad \xi, \eta \ : \ p, q; \quad \phi, \psi,$$

each couple of systems on either side of the colon (:) being cogredient *inter se*: and let F be symbolically represented by

$$(ax+by)(\alpha\xi+\beta\eta)(lp+mq)(\lambda\phi+\mu\psi);$$

then the invariant given by the theorem will be the commutant

$$\begin{array}{l} a\alpha; \quad a\beta+\alpha b; \quad b\beta, \\ l\lambda; \quad l\mu+\lambda m; \quad m\mu. \end{array}$$

The six positions of this are as below written (the first three being positive and the second three negative)

$$\begin{array}{lll} a\alpha;\ a\beta+\alpha b;\ b\beta, & a\alpha;\ a\beta+\alpha b;\ b\beta, & a\alpha;\ a\beta+\alpha b;\ b\beta, \\ l\lambda;\ l\mu+\lambda m;\ m\mu, & l\mu+\lambda m;\ m\mu;\ l\lambda, & m\mu;\ l\lambda;\ l\mu+\lambda m, \\ a\alpha;\ a\beta+\alpha b;\ b\beta, & a\alpha;\ a\beta+\alpha b;\ b\beta, & a\alpha;\ a\beta+\alpha b;\ b\beta, \\ l\mu+\lambda m;\ l\lambda;\ m\mu, & l\lambda;\ m\mu;\ l\mu+\lambda m, & m\mu;\ l\mu+\lambda m;\ l\lambda. \end{array}$$

If we write F under its explicit form,

$$\begin{aligned} &Ax\xi p\phi+Bx\xi p\psi+Cx\xi q\phi+Dx\xi q\psi \\ &+A'x\eta p\phi+B'x\eta p\psi+C'x\eta q\phi+D'x\eta q\psi \\ &+A''y\xi p\phi+B''y\xi p\psi+C''y\xi q\phi+D''y\xi q\psi \\ &+A'''y\eta p\phi+B'''y\eta p\psi+C'''y\eta q\phi+D'''y\eta q\psi, \end{aligned}$$

we have identically the relations following,

$$\begin{array}{llll} a\alpha l\lambda=A, & a\alpha l\mu=B, & a\alpha m\lambda=C, & a\alpha m\mu=D, \\ a\beta l\lambda=A', & a\beta l\mu=B', & a\beta m\lambda=C', & a\beta m\mu=D', \\ b\alpha l\lambda=A'', & b\alpha l\mu=B'', & b\alpha m\lambda=C'', & b\alpha m\mu=D'', \\ b\beta l\lambda=A''', & b\beta l\mu=B''', & b\beta m\lambda=C''', & b\beta m\mu=D''', \end{array}$$

and the commutant expanded becomes

$$\begin{aligned} &A(B'+C''+C'+B'')D'''+(B+C)(D'+D'')A'''+D(A'+A'')(B'''+C''') \\ &-(B+C)(A'+A'')D'''-A(D'+D'')(B'''+C''')-D(B'+C''+C'+B'')A'''. \end{aligned}$$

In the foregoing the x's in the several lines were *for the moment* taken identical, in order the more easily to explain the law of formation of the quantities A. But suppose that they become actually identical for the same line. F then becomes a function of the nth degree in respect to each of p systems of variables, and may be represented symbolically under the form

$$({}^1a^1x + {}^1b^1y + \dots + {}^1l^1t)^n \times ({}^2a^2x + {}^2b^2y + \dots + {}^2l^2t)^n$$
$$\dots \times ({}^pa^px + {}^pb^py + \dots + {}^pl^pt)^n.$$

We may still further limit the generality of the theorem by supposing

$${}^1x = {}^2x = \dots {}^px = x,$$
$${}^1y = {}^2y = \dots {}^py = y,$$
$$\dots\dots\dots\dots\dots\dots$$
$${}^1t = {}^2t = \dots {}^pt = t;$$

F then becomes $$(ax + by + \dots + lt)^{np}.$$

Accordingly, as many different factors as can be found contained an even number of times in the exponent of the function, so many invariants can be formed immediately from a function of any number of variables m by the method of total commutation.

If one of these factors be called n, the commutant corresponding thereto will be of the order

$$\frac{(n+1)(n+2)\dots(n+m-1)}{1.2\dots(m-1)}$$

in respect to the coefficients. Thus take $m = 2$, so that

$$F = (ax + by)^{np}.$$

The general form of such a commutant will be found by taking A_1, $A_2 \dots A_{n+1}$ the coefficients of the several combinations of x, y in $(ax+by)^n$, from which the numerical coefficients n, $\frac{1}{2}n(n-1)$, &c. may be rejected, as only introducing a numerical factor into the result; the commutant will therefore be expressed by means of the form

$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \dots;\ b^n, \qquad (1)$$
$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \dots;\ b^n, \qquad (2)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \dots;\ b^n. \qquad (p)$$

If $p = 2$, the compound commutant

$$a^n;\ a^{n-1}b;\ \dots;\ b^n,$$
$$a^n;\ a^{n-1}b;\ \dots;\ b^n,$$

will easily be seen to be only another form for the catalecticant of $(ax+by)^{2n}$. Thus, let $n=2$,

$$(ax+by)^4 = Ax^4 + 4Bx^3y + 6Cx^2y^2 + 4Dxy^3 + Ey^4\,;$$

so that $$a^4 = A,\quad a^3b = B,\quad a^2b^2 = C,\quad ab^3 = D,\quad b^4 = E.$$

The commutant (which is of the form of the matrix to an ordinary determinant, with the exception that the umbræ enter compoundly instead of simply into the several terms separated by the marks of punctuation), will be

$$\begin{matrix} a^2; & ab; & b^2, \\ a^2; & ab; & b^2: \end{matrix}$$

this, written in the six forms

$$\left.\begin{matrix} a^2; & ab; & b^2 \\ a^2; & ab; & b^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ a^2; & b^2; & ab \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ ab; & a^2; & b^2 \end{matrix}\right\}$$

$$\left.\begin{matrix} a^2; & ab; & b^2 \\ b^2; & ab; & a^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ ab; & b^2; & a^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ b^2; & a^2; & ab \end{matrix}\right\}$$

gives the expression

$$a^4 \times a^2b^2 \times b^4 - a^4 \times (ab^3)^2 - b^4 \times (a^3b)^2 - (a^2b^2)^3 + 2a^3b \times ab^3 \times a^2b^2\,;$$

that is $$ACE - AD^2 - EB^2 - C^3 + 2BCD.$$

One important observation may here be made of a fact which otherwise might easily escape attention, which is, that commutants, where the same terms simple or compound are found in all or several of the lines, in general give rise to products, some of them equal and with the same sign, and others equal but with the *contrary* sign.

This last phenomenon does not manifest itself in commutants appertaining to functions of two variables of the two particular and different species which first and most naturally present themselves, namely where there are only two lines or only two columns*—I believe that it displays itself in every other case of commutantives to functions of two variables. Thus it is that algebraical expressions derived from given functions disguise their symmetry; to make which come to light it becomes necessary to add terms of contrary sign to such expressions. As an example, the reader is invited to develope the cubic invariant of a function of x and y, symbolically expressed by $(ax+by)^8$, where

$$a^8 = A,\quad a^7b = B \ldots ab^7 = H,\quad b^8 = I,$$

* These commutants give respectively the quadrinvariant and the catalecticant, the former of which alone was formerly recognised by Mr Cayley as a commutant.

by means of the commutant

$$\begin{matrix} a^2, & ab, & b^2, \\ a^2, & ab, & b^2, \\ a^2, & ab, & b^2, \\ a^2, & ab, & b^2.\text{*} \end{matrix}$$

Suppose F to be the general even-degreed function of two variables of the degree $2np$.

Let
$$H = \left(\xi \frac{d}{dy} - \eta \frac{d}{dx}\right)^{np} F + \lambda (x\xi + y\eta)^{np},$$

and express H umbrally under the form

$$(ax + by)^{np} (\alpha\xi + \beta\eta)^{np}.$$

* [See p. 346 below.] The number of terms resulting from the independent permutation of each of the 3 linear lines is 6^3, that is 216; but the actual result is (using small letters instead of large) $P - Q$, where

$$P = aei + 3ag^2 + 12beh + 3c^2i + 24cf^2 + 24d^2g + 15e^3,$$
$$Q = 4afh + 4bid + 8bgf + 22ceg + 8chd + 36def,$$

so that the effective number of permutations is only 164. The difference between this and 216 divided by 216 may be termed the Index of Demolition, which we see in this case is $\frac{52}{216}$ or $\frac{13}{54}$; that is, somewhat less than $\frac{1}{4}$. For the cubic invariant of the function of the fourth degree this index is zero, all the permutations being effective. If we take the cubic invariant of the function $ax^{12} + 12bx^{11}y + 66cx^{10}y^2 + \&c. + my^{12}$ under the form $P - Q$, we shall find

$$P = 6ahl + 10ajj + 6bfm + 54bhk + 54cfl + 155cii + 10ddm + 430dgj$$
$$+ 155eek + 520ehh + 520ffl + 280ggg,$$
$$Q = agm + 15aik + 30bgl + 50bij + 15cem + 4cgk + 150chj + 30del + 210dfk$$
$$+ 250dhi + 230efj + 555egl + 660fgh.$$

The number of terms in P and Q is of course the same, and will be found to be 2200 for each; so that out of the 6^5, that is 7776 permutations of the 5 lower rows, only 4400 are effective, and the index of demolition becomes $\frac{3300}{7776}$, that is $\frac{275}{648}$, or rather greater than $\frac{5}{12}$. The Index of Demolition thus goes on constantly increasing as the degree of the function rises; probably (?) it converges either towards $\frac{1}{2}$ or else towards unity. In arranging the terms it will be found most convenient to adopt, as I have done above, the dictionary method of sequence. The computations are greatly facilitated by the circumstance of the effect of any arrangement of each of the 5 lower lines not being altered when these *lines* are permuted with one another; this gives rise to the subdivision of the 7776 permutations into groups as follows: 6 of 120 identical terms, 60 of 60, 36 of 20, 60 of 30, 24 of 20, 30 of 10, 30 of 5, and 6 of 1. So that the total number of permutational arrangements to be constructed is only 252. Other methods of abridging the labour will readily suggest themselves to the practical computer. The total number of the *groups* of terms is of course always known *à priori*, and, for instance, in the case before us, must be equal to the number of ways in which $\frac{1}{2}(12 \times 3)$, that is the number 18, can be divided into 3 parts, none of which is to exceed the number 12, that is 25; for the cubic invariant of the function of the eighth degree of two variables it is the number of ways in which 12 can be divided into 3 parts, of which none shall exceed 8, and so forth, zeros being always understood to be admissible; and of course in general for an invariant of the order r to a function of the degree n of i variables, the number of distinct terms is in general the number of ways in which $\frac{nr}{i}$ can be divided into r parts, of which none shall exceed n, subject however always to the possibility in particular cases of a diminution in consequence of some of the groups assuming zero for their coefficient.

The commutant

$$\begin{array}{ll} a^n, & a^{n-1}b \ldots b^n, \qquad (1) \\ a^n, & a^{n-1}b \ldots b^n, \qquad (2) \\ \multicolumn{2}{l}{\ldots\ldots\ldots\ldots\ldots\ldots} \\ a^n, & a^{n-1}b \ldots b^n, \qquad (p) \\ \alpha^n, & \alpha^{n-1}\beta \ldots \beta^n, \qquad (1) \\ \alpha^n, & \alpha^{n-1}\beta \ldots \beta^n, \qquad (2) \\ \multicolumn{2}{l}{\ldots\ldots\ldots\ldots\ldots\ldots} \\ \alpha^n, & \alpha^{n-1}\beta \ldots \beta^n, \qquad (p) \end{array}$$

will be a function of λ, and all the several coefficients will be invariants of F*.

When $p = 1$ we obtain the Λ given in the preceding section, and originally published by me in the *Philosophical Magazine* for the month of November, 1851. The Λ obtained on this supposition has for its coefficients a series of independent invariants, commencing with the catalecticant and closing with the quadratic invariant. When p has any other value, we observe a similar series commencing with a commutantive invariant of a lower order than the catalecticant, but always closing with the quadratic invariant. Thus, for example, when $2np=8$, we may obtain by the preceding theorem three different quadratic functions; one giving the invariants of the orders 5, 4, 3, 2, the second those of the orders 3, 2, the third the invariant of the order 2.

In this case the invariants of the same order given by the different Λ's are the same to numerical factors *près*. Whether this is always necessarily the case is a point reserved for further examination.

The commutants applied in the preceding theorems have been called by me total commutants, because the total of each line of umbræ is permuted in every possible manner. If the lines be divided into segments, and the permutation be local for each segment instead of extending itself over the whole line, we then arrive at the notion of partial commutants, to which I have also (in concert with Mr Cayley) given the distinctive name of Intermutants. In order to find the invariants of functions of odd degrees, the theory of total commutants requires the process of commutation to be applied, not immediately to the coefficients of the proposed function, but to some derived concomitant form. I became early sensible of this imperfection, and stated to the friend above named, to whom I had previously

* By substituting the symbols $\frac{d}{dx}$, $\frac{d}{dy}$, &c. in place of the *umbræ* a, b, &c., the theorem is easily stated for covariants generally. But in applying the commutantive method to obtain covariants, or rather in the statement of the results flowing from each application, it is never necessary to go beyond the case of invariants, because the commutantive covariants of any given homogeneous function are always identical with commutantive invariants of emanants of the same function.

imparted my general method of total commutation, my conviction of the existence of a qualified or restricted method of permutation, whereby the invariants of the cubic function, for instance, of two and of three letters would admit, without the aid of a derived form, of being represented. Many months ago, when I was engaged in this important research, and had made some considerable steps towards the representation of the invariant, that is, the discriminant of the cubic function of x and y, under the form of a single permutant, I was surprised by a note from the friend above alluded to, announcing that he had succeeded in fixing the form of the permutant of which I was at that moment in search. It is with no intention of complaining of this interference on the part of one to whose example and conversation I feel so deeply indebted, (and the undisputed author of the theory of Invariants,) that I may be permitted to say that, independent of the intervention of this communication, I must inevitably have succeeded in shaping my method so as to furnish the form in question; and that with greater certainty, after my theory of commutants had furnished me with the precedent of permutable forms giving rise to terms identical in value but affected with contrary signs. As I have understood that Mr Cayley is likely to develop this part of the subject in the present number of the *Journal*, it will be the less necessary for me to enter at any length into the theory of partial commutants on the present occasion.

The method of partial commutation is a simple but most important corollary from that of total commutation hereinbefore explained. To fix the ideas, conceive a class of p cogredient systems, and that there are qr such classes perfectly independent. Proceed to divide these qr classes in any manner whatever into r sets, each containing q classes; and form the symbol of the total commutant corresponding to each such set. Now let these commutants be placed side by side against one another, and transpose the terms in each compound line thus formed once for all, but in any arbitrary manner. Then permute in every possible way all those symbols in each line, *inter se*, which belong to the same class, and operate with the symbols thus produced by reading off the vertical columns and attending to the rule of the + and − signs, as in the case of a total commutant; the result will be a commutant of the form operated upon. For instance, let $p=1$, $q=3$, $r=2$, and let the number of variables in each system be 2. Form the commutant operators

$$\begin{array}{cc|cc}
\frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{d\xi}, & \frac{d}{d\eta}, \\
\frac{d}{dp}, & \frac{d}{dt}, & \frac{d}{d\phi}, & \frac{d}{d\theta}, \\
\frac{d}{dr}, & \frac{d}{ds}, & \frac{d}{d\rho}, & \frac{d}{d\sigma}.
\end{array}$$

Interchange *in any manner* but *once for all* the symbols in each line, as thus:

$$\frac{d}{dx}, \ \frac{d}{dy}, \ \frac{d}{d\xi}, \ \frac{d}{d\eta},$$
$$\frac{d}{d\phi}, \ \frac{d}{dp}, \ \frac{d}{dt}, \ \frac{d}{d\theta},$$
$$\frac{d}{ds}, \ \frac{d}{d\rho}, \ \frac{d}{dr}, \ \frac{d}{d\sigma}.$$

Now permute, *inter se*, the variables of each system, as

$$\frac{d}{dx}, \ \frac{d}{dy}; \ \frac{d}{dp}, \ \frac{d}{dt}, \ \text{\&c.};$$

the total number of the operative forms resulting will be $(1 \,.\, 2)^6$, and the sum of the $(1 \,.\, 2)^6$ quantities, half positive and half negative, formed after the type of

$$\Sigma \left\{ \begin{array}{c} \dfrac{d}{dx}\dfrac{d}{d\phi}\dfrac{d}{ds}U \times \dfrac{d}{dy}\dfrac{d}{dp}\dfrac{d}{d\rho}U \\ \times \dfrac{d}{d\xi}\dfrac{d}{d\theta}\dfrac{d}{dr}U \times \dfrac{d}{d\eta}\dfrac{d}{dt}\dfrac{d}{d\sigma}U \end{array} \right\},$$

U being supposed to be a function homogeneous in

$$x, y; \ \xi, \eta; \ p, t; \ \phi, \theta; \ r, s; \ \rho, \sigma,$$

will be a covariant of U.

The proof of the truth of this proposition is contained in what is shown in the Notes of the Appendix for total commutants, it being only necessary to make the systems which are independent vary consecutively, and then apply the inference to the supposition of their varying simultaneously.

It may be extended to the more general supposition of classes of an unequal number of cogredient systems of unequal numbers of variables in each, the only condition apparently required being that the number of distinct terms shall be the same in each line of the final commutantive operator. The important remark to be made is, that in applying this theorem there is nothing to prevent any of the systems being made *identical*; or, in other words, a given function of one system of variables may be regarded as a function of as many different, although coincident, sets as we may choose to suppose. Thus, suppose

$$U = Ax^2 + 2Bxy + Cy^2,$$

we may take the partial commutant formed of the two total commutant operators

$$\frac{d}{dx}, \ \frac{d}{dy},$$
$$\frac{d}{dx}, \ \frac{d}{dy},$$

combined with itself. If we write them in the same order,

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

(where I use the dots and dashes to distinguish those in the same line which are considered as belonging to the same class, and therefore as permutable, *inter se*), we shall evidently obtain $4\{AC - B^2\}^2$; if we commence with a permutation, so as to have the form of operation

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\acute{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\dot{d}}{dy},$$

it will be found that we obtain $2\{AC - B^2\}^2$.

Again, suppose that we have

$$U = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3.$$

If we write

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

the value of the commutant would come out zero; but if we make a permutation, and write

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\acute{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\dot{d}}{dy},$$

the operation indicated by the above performed upon U, will give a multiple of the discriminant of U.

In like manner we may represent Aronhold's Sextic Invariant of the form $(x, y, z)^3$ by means of the partial commutant

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dz}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy}, \frac{\acute{d}}{dz},$$

$$\frac{\acute{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dz}, \frac{\dot{d}}{dx}, \frac{\acute{d}}{dy}, \frac{\acute{d}}{dz},$$

$$\frac{\acute{d}}{dx}, \frac{\acute{d}}{dy}, \frac{\dot{d}}{dz}, \frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dz}.$$

If we make

$$V = \left(\xi' \frac{d}{d\xi} + \eta' \frac{d}{d\eta} + \zeta' \frac{d}{d\zeta}\right) \left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz}\right)^2 (x, y, z)^3,$$

and use H to signify the determinant

$$\begin{vmatrix} x, & y, & z \\ \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \end{vmatrix},$$

which is evidently an universal triple covariant, and make

$$W = V + \lambda H,$$

and apply to W the partial commutantive symbol

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dz}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy}, \frac{\acute{d}}{dz},$$

$$\frac{\acute{d}}{d\xi}, \frac{\dot{d}}{d\eta}, \frac{\dot{d}}{d\zeta}, \frac{\dot{d}}{d\xi}, \frac{\acute{d}}{d\eta}, \frac{\acute{d}}{d\zeta},$$

$$\frac{\acute{d}}{d\xi'}, \frac{\acute{d}}{d\eta'}, \frac{\dot{d}}{d\zeta'}, \frac{\dot{d}}{d\xi'}, \frac{\dot{d}}{d\eta'}, \frac{\acute{d}}{d\zeta'},$$

we shall obtain a function of λ of which all the odd powers and the second power will disappear, and such that the coefficients of λ^2 and the constant term will be Aronhold's S and T, and the discriminant of the entire function in respect to λ^2 (if not for the distribution assigned to the dots and dashes in the foregoing, at least for some other distribution) may not improbably be the discriminant of the given function $(x, y, z)^3$.

NOTES IN APPENDIX.

(1) [p. 295 above.] More generally, in as many ways as the number n can be divided into parts, in so many ways can a given function of one set of variables be as it were *unravelled* so as to furnish concomitant forms.

For instance, the form $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ has for a concomitant

$$aux + buy + bvx + cvy + cwx + dwy,$$

where u, v, w are cogredient with $x^2, 2xy, y^2$; and also

$$auu'x + buu'y + buv'x + bvu'x + cvv'x + cvu'y + cuv'y + dvv'y,$$

where $u, v; u', v'$ are cogredient with each other and with x and y; and the proposition in the text may be best derived from this more general theorem by dividing the index into equal parts, forming as many systems as there are such parts, and then identifying the systems so formed.

(2) [p. 297 above.] The following additional example will illustrate the power of this method.

Let $\phi = (x, y, z)^4$ be the general function of the fourth degree. Form by *unravelment* the concomitant form $(u, v, w, p, q, r)^2$ (say P) where u, v, w, p, q, r are cogredient with $x^2, y^2, z^2, 2zy, 2xz, 2yx$.

Again, the universal concomitant $(x\xi + y\eta + z\zeta)^2$ will have for its concomitant

$$u\xi^2 + v\eta^2 + w\zeta^2 + p\eta\zeta + q\zeta\xi + r\xi\eta,$$

where ξ, η, ζ are contragredient to x, y, z. Now take the reciprocal polar of this last form with respect to ξ, η, ζ; that is,

$$\Sigma\,(vw - \tfrac{1}{4}p^2)\,x_1{}^2 + 2\Sigma\,(\tfrac{1}{4}qr - \tfrac{1}{2}pu)\,y_1z_1 \text{ (say } G),$$

where x_1, y_1, z_1, being contragredient to ξ, η, ζ, will be cogredient with x, y, z. $P + \lambda G$ is a quadratic function of the six variables u, v, w, p, q, r, and its discriminant will give a function of λ of the sixth degree, all of whose even coefficients will be covariants of ϕ. If we replace x_1, y_1, z_1 by x, y, z, these even coefficients will be respectively (understanding that *order* refers to the dimensions *quoad* the coefficients of ϕ and *degree* to the dimensions *quoad* x, y, z) as follows:

Of order	6	degree	0,
„	5	„	2,
„	4	„	4,
„	3	„	6,
„	2	„	8,
„	1	„	10,
„	0	„	12.

The two last coefficients must evidently be identically zero. It is *possible* that some of the others may be so too: as regards the one of the third order and sixth degree, this is of the same form as, and may be identical with, the Hessian of ϕ; as regards the one of the fourth order and fourth degree, this may be ϕ itself multiplied by the cubic invariant (which the theory of Section III. proves to exist) of ϕ. But the covariants of the fifth order and second degree, and of the second order and eighth degree, if they are not identically zero, and if the latter is not ϕ^2 (which a trial or two of some very simple cases will easily establish one way or the other) are probably irreducible forms. The existence of a correlated conic section to a curve of the fourth order, if established, would be particularly interesting, and its geometrical meaning would well deserve being elicited.

(3) [p. 303 above.] If any form (f) of the degree n be written symbolically,

$$(a_1x_1 + a_2x_2 + \ldots + a_\iota x_\iota)^n,$$

where $x_1, x_2 \ldots x_\iota$ are real but $a_1, a_2 \ldots a_\iota$ umbral, and if I_r be any invariant of the order r in respect of the real coefficients of (f), it is easily seen by reason of I_r remaining unaltered when $x_1, x_2 \ldots x_\iota$ become respectively $f_1x_1, f_2x_2 \ldots f_\iota x_\iota$, provided that $f_1, f_2 \ldots f_\iota = 1$, that each term in I_r expressed by means of the umbræ, must contain an equal number of times $a_1, a_2 \ldots a_\iota$, so that each such term will contain $\frac{nr}{\iota}$ of each of them, of course differently subdivided and grouped; hence we have the universal condition that $\frac{nr}{\iota}$ must be an integer; but this is less stringent than the actual condition, which is that $\frac{nr}{\iota}$ must be an integer of a certain form; for instance, as before observed, when $\iota = 2$, $\frac{nr}{\iota}$ must be an *even* integer.

(4) [p. 307 above.] To prove the theorem given in the text for total simple commutants it is only necessary to bear in mind that whenever two columns in any total commutant become identical, the commutant vanishes. To fix the ideas, take the commutant formed of lines similar to $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, written

under one another; let there be r such lines, the total number of terms will be $(1.2.3)^r$: the $1.2.3$ positions of the line written above will correspond to $(1.2.3)^{r-1}$ several groupings of the remaining lines. Now when x, y, z undergo a unimodular linear substitution, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will undergo a related substitution not coincident with that of x, y, z, but still unimodular; let x, y, z change, all the other systems remaining fixed, and suppose $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ to become respectively

$$f\frac{d}{dx}+g\frac{d}{dy}+h\frac{d}{dz},$$

$$f'\frac{d}{dx}+g'\frac{d}{dy}+h'\frac{d}{dz},$$

$$f''\frac{d}{dx}+g''\frac{d}{dy}+h''\frac{d}{dz},$$

then each of the $(1.2.3)^{r-1}$ groups of the terms arising from the permutation of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will subdivide into 27 groups, of which we may reject those in which any of the terms $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$ occurs twice or three times; accordingly there will be left only the six effective orders of permutations,

$$\left(f\frac{d}{dx}, g'\frac{d}{dy}, h''\frac{d}{dz}\right); \left(f\frac{d}{dx}, h'\frac{d}{dz}, g''\frac{d}{dy}\right); \text{ \&c.}$$

consequently each of the $(1.2.3)^{r-1}$ groups gives rise to 6 times 6 products whose sum will be $\begin{vmatrix} f'', & g'', & h'' \\ f', & g', & h' \\ f, & g, & h \end{vmatrix}$ × the sum of the 6 products corresponding to the permutations of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$; and therefore, the transformations being unimodular, the sum of the products corresponding to the entire $(1.2.3)^r$ permutations remains constant when x, y, z change. In like manner, all the systems may change one after the other, and consequently all of them at the same time without affecting the value of the commutant: and in like manner for the general case. Q.E.D.

(5) [p. 312 above.] The truth of the proposition relative to compound commutants and the mode of the demonstration will be apparent from the subjoined example.

Let the function be supposed to be

$$(ax+by)(a'x'+b'y')(\alpha\xi+\beta\eta)(\alpha'\xi'+\beta'\eta'),$$

where x, y; x', y' are cogredient and ξ, η; ξ', η' cogredient; the a, b, α, β, &c. are of course mere umbræ. Now take the compound commutant

$$\begin{matrix} aa', & ab' + a'b, & bb', \\ \alpha\alpha', & \alpha\beta' + \alpha'\beta, & \beta\beta'. \end{matrix}$$

Let x, y; x', y' undergo a linear substitution, and, accordingly,

$$\begin{array}{lcl} \text{let } a & \text{become} & fa + gb, \\ a' & ,, & fa' + gb', \\ b & ,, & ha + kb, \\ b' & ,, & ha' + kb', \end{array}$$

f, g, h, k being of course actual and not umbral; then the above commutant will be easily seen to decompose into 6 others, which will be equal to the original commutant multiplied by the determinant

$$\begin{vmatrix} f^2, & 2fg, & g^2 \\ fh, & fk + gh, & gk \\ h^2, & 2hk, & k^2 \end{vmatrix},$$

which is equal to $(fk - gh)^3$, that is $= 1$.

And so in general, which shows, as in the preceding note, that all the classes of cogredient systems may be transformed successively one after the other, and therefore simultaneously, without altering the value of the commutant.

(6) In the last May Number* of the *Journal*, Mr Boole, to whose modest labours the subject is perhaps at least as much indebted as to any one other writer, has given a theorem†, (14) p. 94, the excellent idea contained in which there is no difficulty in shaping so as to render it generalizable by aid of the theory of contravariants. It may be regarded in some sort a pendant or reciprocal to the Eisenstein-Hermite theorem, presented by me under a wider aspect in the First Section of this paper.

[* *Camb. and Dub. Math. Journ.* Vol. VI. (1851), pp. 87—106.]

† Mr Boole applied his theorem to obtain the cubic invariant of $(x, y)^4$, say $\phi(x, y)$, by operating upon its Hessian with $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right)$. More generally, when $\phi(x, y) = (x, y)^{2n}$, the catalecticant of the antepenultimate emanant of ϕ is also of the degree $2n$; and this, when operated upon by $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right)$, will give an invariant of the order $n+1$, which is probably identical with the catalecticant of ϕ itself. There exists a most interesting transformation of the catalecticant of any emanant of a function of any degree in x, y, whether even or odd, under the form of a determinant some of the lines of which contain combinations only of x and y, without any of the coefficients, and all the rest the coefficients only of the given function without x or y. The Hessian being the catalecticant of the second emanant is of course included within this statement.

Let $\phi(x, y \ldots z)$ have any contravariant $\theta(x, y \ldots z)$; then will

$$\phi\left(\frac{d}{dx}, \frac{d}{dy} \cdots\cdots \frac{d}{dz}\right) . \theta(x, y \ldots z)$$

be a contravariant of ϕ. For orthogonal transformations the terms contravariant and covariant coincide, and the theorem for this case appears to have been known to Mr Boole, see (15), same page. More generally, if ψ and θ be any two concomitants of ϕ, the algebraical product $\psi\theta$ will also be a concomitant of ϕ, provided that the systems of variables in ψ and θ have all distinct names, or that those which bear the same names are cogredient with one another. If this proviso does not hold good, the product in question will evidently be no longer a concomitant of ϕ. Let however Ψ denote what ψ becomes, and ϑ what θ becomes, when in place of the variables $x, y \ldots z$ of every two contragredient synonymous systems in ψ and θ we write $\frac{d}{dx}, \frac{d}{dy} \cdots\cdots \frac{d}{dz}$, then will $\vartheta\psi$ and $\Psi\theta$ be each of them concomitants of ϕ, the synonymous systems becoming cogredient with ψ in the one case and with θ in the other.

(7) There is one principle of *paramount* importance which has not been touched upon in the preceding pages, which I am very far from supposing to exhaust the fundamental conceptions of the subject, (indeed, not to name other points of enquiry, I have reason to suppose that the idea of contragredience itself admits of indefinite extension through the medium of the reciprocal properties of commutants; the particular kind of contragredience hereinbefore considered having reference to the reciprocal properties of ordinary determinants only).

The principle now in question consists in introducing the idea of *continuous* or *infinitesimal* variation into the theory. To fix the ideas, suppose C to be a function of the coefficients of $\phi(x, y, z)$, such that it remains unaltered when x, y, z become respectively fx, gy, hz, provided that $fgh = 1$. Next, suppose that C does not alter when x becomes $x + ey + \epsilon z$, when e and ϵ are indefinitely small: it is easily and obviously demonstrable that if this be true for e and ϵ indefinitely small, it must be true for *all* values of e and ϵ. Again, suppose that C alters neither when x receives such an infinitesimal increment, y and z remaining constant, nor when y nor z separately receive corresponding increments, z, x and x, y in the respective cases remaining constant; it then follows from what has been stated above that this remains true for finite increments to x or y or z separately; and hence it may easily be shown that C will remain constant for any *concurrent* linear transformations of x, y, z, when the modulus is unity. This all-important principle enables us at once to fix the form of the symmetrical functions of the roots of $\phi\left(\frac{x}{y}, 1\right)$ which represent invariants of $\phi(x, y)$ when the coefficient of the highest power of x is made unity. It also *instantaneously* gives the necessary and *sufficient* conditions to which an invariant of any given order of any homogeneous function whatever is subject, and thereby reduces the problem of discovering invariants to a definite form. But as these conditions coincide with those which have been stated to me as derived from other considerations by the gentleman whose labours in this department are *concomitant* with my own, I feel myself bound to abstain from pressing my conclusions until he has given his results to the press.

(8) By aid of the general principle enunciated in Note (6) above, we can easily obtain Aronhold's S and T. Let U be the given cubic function of x, y, z, and let $G(x, y, z;\ \xi, \eta, \zeta)$ be the polar reciprocal in respect to ξ, η, ζ of $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz}\right)^2 U$, then $G(\xi, \eta, \zeta;\ x, y, z)$ as well as the former G will be a concomitant to U, but the homonymous systems of variables in the two G's will be contragredient; and, accordingly,

$$G\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz};\ \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}\right) . G(\xi, \eta, \zeta;\ x, y, z)$$

will be a concomitant to U; this concomitant is readily seen to be an invariant of the fourth order; that is, Aronhold's S. Again, from S, by means of the Eisenstein-Hermite theorem, we may derive a form $K(x, y, z)$ of the third degree in x, y, z, and whose coefficients will be of three dimensions; and, accordingly, if the Hessian of U be called $H(U)$,

$$K\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right) . H(U)$$

will be a Sextic Invariant of U, that is, Aronhold's T.