43.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

PART I. SECTION IV. *Reciprocity, also Properties and Analogies of certain Invariants, &c.*

IT will hereafter be found extremely convenient to represent all systems of variables cogredient with the original system in the primitive form by letters of the Roman, and all contragredient systems by letters of the Greek alphabet; the rules for concomitance may then be applied without paying any regard to the distinction between the direction of the march of the substitutions, the variables at the close of each operation as it were telling their own tale in respect of being cogredients or contragredients. This distinction has not (as it should have) been uniformly observed in the preceding sections; as, for instance, in the notation for emanants which have been derived by the application of the symbol $\left(\xi \dfrac{d}{dx} + \eta \dfrac{d}{dy} + \&c.\right)^2$, instead of the more appropriate one $\left(x' \dfrac{d}{dx} + y' \dfrac{d}{dy} + \&c.\right)^2$.

The observations in this section will refer exclusively to points of doctrine which have been started in the preceding sections in such order as they more readily happen to present themselves. And, first, as to some important applications of the reciprocity method referred to in Notes (6) and (8) of the Appendix [pp. 325, 327 above].

The practical application of this method will be found greatly facilitated by the rule that x, y, z, &c. may always in any combination of concomitants be replaced respectively by $\dfrac{d}{d\xi}$, $\dfrac{d}{d\eta}$, $\dfrac{d}{d\zeta}$, &c., and *vice versâ*. I shall apply this prolific principle of reciprocity to elucidate some of the properties and relations of Aronhold's S and T, and certain other kindred forms. This S and T are the quartinvariant and sextinvariant respectively of a cubic of three variables. I give the names of s and t to the quadrinvariant and cubinvariant of the quartic function of two variables. Furthermore, whoever will consider attentively the remarks made in Section II. of the foregoing relative to reciprocal polars, will apprehend without any difficulty that to every invariant of a function of any degree of any number of variables will

correspond a contravariant of a function of the same degree of variables one more in number, and that between such invariants, whatever relations exist expressed independently of all other quantities, precisely the same relations must exist between the corresponding contravariants. Thus, then, to s and t the two invariants of $(x, y)^4$ will correspond two contravariants σ and τ of $(x, y, z)^4$, and to S and T the two invariants of $(x, y, z)^3$ will correspond Σ and ϑ two contravariants of $(x, y, z, t)^3$. Calling r the resultant of $(x, y)^4$, R the resultant of $(x, y, z)^3$, ρ the polar reciprocal, or, more briefly, the reciprocant of $(x, y, z)^4$, and (R) the reciprocant of $(x, y, z, t)^3$, we have the following equations (presuming that all the quantities are previously affected with the proper numerical multipliers), namely

$$r = s^3 + t^2, \qquad \rho = \sigma^3 + \tau^2,$$
$$R = S^3 + T^2, \quad (R) = \Sigma^3 + \vartheta^2.$$

I propose in this First Annotation to point out the remarkable analogies which exist between the modes of generating the four pairs of quantities s, t, &c., the functions severally corresponding to which I shall call u, ω, U, Ω. The Hessian corresponding to any of these functions will be denoted by an H prefixed, and when we have to consider, not the pure Hessian, but the matrix formed from it by adding a vertical and horizontal border of variables, the same in number but contragredient to the variable of the function (as, for instance, the Hessian of u bordered with ξ, η horizontally and vertically, or of U with ξ, η, ζ), then I shall denote the result by the ruled symbol $\bar{H}$, and if there be occasion to add two borders, as ξ, η, ζ; ξ', η', ζ', both repeated in the horizontal and vertical directions, the result will be typified by the doubly ruled $\bar{\bar{H}}$.

Now, in the first place, as observed by me in Note (8) of the Appendix in the last number; if we call the coefficients of U (10 in number) a, b, c, d, &c., we have

$$S = \bar{H} \left\{ \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\} \bar{H} \{x, y, z; \xi, \eta, \zeta\},$$

also

$$T = \frac{dS}{da} \frac{d^3H}{dx^3} + \frac{dS}{db} \frac{d^3H}{d^2xdy} + \frac{dS}{dc} \frac{d^3H}{d^2xdz} + \&c.$$

I will now add the further important relation

$$S^2 = \frac{dT}{da} \frac{d^3H}{dx^3} + \frac{dT}{db} \frac{d^3H}{d^2xdy} + \frac{dT}{dc} \frac{d^3H}{d^2xdz} + \&c.^*,$$

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* It will be found hereafter convenient to designate contravariants formed in this manner from invariants as *Evects* of such invariants or contravariants, and according to the number of times that such process of derivation is applied, 1st, 2nd, 3rd, &c. evects. Such evects form a peculiar class, and when considered generally, without reference to the base to which they refer, they may be termed evectants. Evectants will be again distinguishable according as their base is an invariant simply or a contravariant. Perhaps the terms pure and affected evectants may serve to mark this distinction.

so that it will be observed if all the derivatives of S are zero, T is zero, and *vice versâ*.

Precisely in the same way, using h and $\bar{h}$ to denote respectively the Hessian of u and the same bordered with ξ, η, we have

$$s = \bar{h}\left(\frac{d}{d\xi}, \frac{d}{d\eta}; \frac{d}{dx}, \frac{d}{dy}\right)\bar{h}(x, y; \xi, \eta),$$

$$t = \frac{ds}{da}\frac{d^4h}{dx^4} + \frac{ds}{db}\frac{d^4h}{dx^3 dy} + \frac{ds}{dc}\frac{d^4h}{dx^2 dy^2} + \&c.$$

$$s^2 = \frac{dt}{da}\frac{d^4h}{dx^4} + \frac{dt}{db}\frac{d^4h}{dx^3 dy} + \frac{dt}{dc}\frac{d^4h}{dx^2 dy^2} + \&c.$$

Again, taking $(\overline{\overline{H}})$ the second bordered Hessian of Ω; that is, Ω bordered as well horizontally as vertically with the double lines and columns ξ, η, ζ, θ; ξ', η', ζ', θ',

$$\Sigma = (\overline{\overline{H}})\left(\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}, \frac{d}{d\theta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{dt}; \xi', \eta', \zeta', \theta'\right)$$

$$\times (\overline{\overline{H}})(x, y, z, t; \xi, \eta, \zeta, \theta; \xi', \eta', \zeta', \theta'),$$

$$\vartheta = \frac{d\Sigma}{da}\frac{d^3\bar{H}}{dx^3} + \frac{d\Sigma}{db}\frac{d^3\bar{H}}{dx^2 dy} + \frac{d\Sigma}{dc}\frac{d^3\bar{H}}{dx^2 dz} + \frac{d\Sigma}{dd}\frac{d^3\bar{H}}{dx^2 dt} + \&c.,$$

$$\Sigma^2 = \frac{d\vartheta}{da}\frac{d^3\bar{H}}{dx^3} + \frac{d\vartheta}{db}\frac{d^3\bar{H}}{dx^2 dy} + \&c.$$

In like manner again

$$\sigma = (\overline{\overline{h}})\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \xi', \eta', \zeta'\right\}$$

$$\times \bar{h}\{x, y, z; \xi, \eta, \zeta; \xi', \eta', \zeta'\},$$

$$\tau = \frac{d\sigma}{da}\frac{d^4(\bar{h})}{dx^4} + \&c.,$$

$$\sigma^2 = \frac{d\tau}{da}\frac{d^4\bar{h}}{dx^4} + \&c.,$$

σ and τ are the same quantities as are calculated by Mr Salmon, in his inestimable work *On Higher Plane Curves*, but are there expressed under the names of S and T, with the sole difference that in place of x, y, z, used by Mr Salmon, the contragredient variables ξ', η', ζ' are used in the expressions above. Mr Salmon has also pointed out to me that σ may be obtained by operating with

$$\left(\xi^4 \frac{d}{da} + \xi^3\eta \frac{d}{db} + \xi^3\zeta \frac{d}{dc} + \&c.\right)$$

directly upon I a cubic invariant of the function u, or $(x, y, z)^4$. This I is no other than the simple commutant obtained by operating upon u with the commutantive symbol formed by taking four times over the line $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz},$ agreeable to the remark made in the third section that

every function of an even degree of n variables possesses an invariant of the nth order in extension of Mr Cayley's observation that every such function of two variables possesses a quadrinvariant, that is an invariant of the second order.

I need hardly remark that σ is of 2 dimensions in the coefficients and of 4 in the contragredient variables, τ of 3 in the coefficients and of 5 in the contragredients, Σ of 4 in the constants and 4 in the contragredients, ϑ of 6 in the constants and 6 in the contragredients, or that the single-bordered Hessians of u and U and the double-bordered Hessians of ω and Ω are each of them quadratic in respect of the x &c. as well as of the ξ &c. systems.

If the right numerical factors be attributed to S, T, Aronhold has shown that

$$H\{H(U)\} + T.H(U) + S^2 U = 0,$$

and in my paper in the last May Number*, I gave the equation

$$h\{h(u)\} + s.h(u) + tu = 0.$$

I think it highly probable that it will be found that the analogous equations obtain, namely

$$\bar{H}\{\bar{H}(\Omega)\} + \vartheta.\bar{H}(\Omega) + \Sigma^2\Omega = 0,$$

$$\bar{h}\{\bar{h}(\omega)\} + \sigma.\bar{h}(\omega) + \tau\omega = 0.$$

These remarkable equations, if verified (of which I can scarcely doubt), will be most powerful aids to the dissection of the forms ω, Ω, and thereby to the detection of the fundamental properties of curves of the fourth and surfaces of the third degree, of which at present so little is known. It will have been observed that in the preceding developments the contravariants of ω and Ω were derived in precisely the same way from ω and Ω as the corresponding invariants of u and U from u and U, with the sole difference that the Hessian used in the two latter cases is replaced by a single-bordered Hessian in the two former cases, and a single-bordered Hessian in the two latter by a double-bordered Hessian in the two former. The analogies are not even yet stated exhaustively; for it will be remembered (as shown in the third section), that T and S can be derived directly and concurrently by means of operating with the commutantive symbol

$$\left. \begin{array}{ccc} \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz} \\[2mm] \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz} \\[2mm] \dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta} \\[2mm] \dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta} \end{array} \right\} \text{ upon } \bar{H}(U) + \lambda(x\xi + y\eta + z\zeta)^2,$$

[* p. 192 above.]

which gives a result of the form $m(\lambda^3 + S\lambda + T)$, m being a number; and I conjecture that if

$$\frac{d}{dx}, \quad \frac{d}{dy}, \quad \frac{d}{dz}, \quad \frac{d}{dt},$$

$$\frac{d}{dx}, \quad \frac{d}{dy}, \quad \frac{d}{dz}, \quad \frac{d}{dt},$$

$$\frac{d}{d\xi}, \quad \frac{d}{d\eta}, \quad \frac{d}{d\zeta}, \quad \frac{d}{d\theta},$$

$$\frac{d}{d\xi}, \quad \frac{d}{d\eta}, \quad \frac{d}{d\zeta}, \quad \frac{d}{d\theta},$$

be made to operate upon

$$\overline{\overline{H}}\Omega + \lambda (x\xi + y\eta + z\zeta + t\theta)^2,$$

and the result be put under the form

$$m(\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D),$$

that A will be zero, B and C will be respectively Σ and ϑ, and perhaps D (a contravariant, if it effectively exist, of 8 dimensions in the coefficients of Ω, and of a like number in the contragredients ξ', η', ζ', θ'), also zero. But of the evanescence of D I do not speak with any degree of assurance.

Mr Salmon has made an excellent observation to the effect that if we call (σ) what σ becomes when ξ', η', ζ' are replaced by $\dfrac{d}{dx}$, $\dfrac{d}{dy}$, $\dfrac{d}{dz}$, $(\sigma)\,h(\omega)$ will represent a covariant to ω of $3 + 2$, that is, 5 dimensions in the coefficients, and of $6 - 4$, that is, of 2 dimensions in x, y, z, $h(\omega)$ being of 3 and 6 dimensions in these respectively, and σ of 2 and 4 dimensions respectively in the same. Now these resulting dimensions 5 and 2 precisely agree with the form especially noticed by me in Note* (2) of the Appendix, where it was derived as one of a group by the method of unravelment. There can be little doubt that these two conics each of them indissolubly connected with every curve of the fourth degree are identical. The form $(\sigma)\,h(\omega)$ enables us to prove readily (thanks to Mr Salmon's calculation of σ, given in his *Higher Plane Curves*, under the name of S) that this is a *bonâ fide* existent conic.

For if we take a particular case of ω, say

$$\omega = a_1 x^4 + b_2 y^4 + c_3 z^4 + 6 d y^2 z^2,$$

we find

$$h(\omega) = \begin{vmatrix} a_1 x^2, & 0, & 0 \\ 0, & b_2 y^2 + d z^2, & d y z \\ 0, & d y z, & c_3 z^2 + d y^2 \end{vmatrix}$$

$$= a_1 (b_2 c_3 + d^2)\, x^2 y^2 z^2 + a_1 b_2 d x^2 y^4 + a_1 c_3 d x^2 z^4,$$

[* p. 323 above.]

and σ becomes

$$a_1 d\eta'^2\zeta'^2,$$

and consequently (σ) is

$$a_1 d\left(\frac{d}{dy}\right)^2\left(\frac{d}{dz}\right)^2,$$

and therefore

$$(\sigma)\, h\,(\omega) = 4a_1^2 d\,(b_2 c_3 + d^2)\, x^2,$$

the conic here reducing to a pair of coincident straight lines. This example demonstrates that the conic is in general actually existent.

As I have said so much upon S and T it may not be irrelevant to state in this place how I obtained the conditions for U, the characteristic of the curve of the third degree becoming the characteristic of a conic and a straight line, that is breaking up into a linear and a quadratic factor, which Mr Salmon has inserted in the notes to his work above referred to. When U is of this form it may obviously by linear transformations be expressed by $ax^3 + 6dxyz$, but when starting with the general form,

$$a_1 x^3 + b_2 y^3 + c_3 z^3 + \&c. + 6Dxyz,$$

we form two contravariants from S and T, to wit

$$\left(\xi^3 \frac{d}{da_1} + \eta^3 \frac{d}{db_2} + \zeta^3 \frac{d}{dc_3} + \&c. + \xi\eta\zeta\frac{d}{dD}\right)S, \text{ say } S',$$

$$\left(\xi^3 \frac{d}{da_1} + \eta^3 \frac{d}{db_2} + \zeta^3 \frac{d}{dc_3} + \&c. + \xi\eta\zeta\frac{d}{dD}\right)T, \text{ say } T',$$

and then make $a_1 = a$, $D = d$, and all the other coefficients zero, it will easily be seen on examining the forms of S and T, given by Mr Salmon, that (S) and (T) (the evectants of S and T) become respectively

$$4d^3\xi\eta\zeta, \quad 31d^5\xi\eta\zeta;$$

we have therefore $(T) + \lambda\,(S) = 0$: and (T) and (S), although contravariantive to their primitive U, are covariantive with one another, so that $(T) + \lambda\,(S) = 0$ is a persistent relation unaffected by linear transformations; it follows therefore that when U is of, or reducible to, the form supposed,

$$\frac{dS}{da_1} : \frac{dS}{db_2} : \frac{dS}{dc_3} : \&c. : \frac{dS}{dD}$$

$$= \frac{dT}{da_1} : \frac{dT}{db_2} : \frac{dT}{dc_3} : \&c. : \frac{dT}{dD},$$

which is the criterion given in the note referred to*.

I am also able to obtain these equations more directly by another method founded upon a New View of the Theory of Elimination, an account of which,

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* Mr Salmon has remarked that the two evectants (S) and (T) intersect in the nine cuspidal points of the polar reciprocal to the curve.

however, I must reserve for another occasion, but which, I may mention, serves to fix not merely the conditions, as in the ordinary restricted theory, that a given set of equations may be simultaneously satisfiable by some one system of values of the variables, but the *conditions* that such set of equations may be simultaneously satisfiable by any given number of distinct systems of variables.

Mr Salmon has remarked to me to the effect that if in τ we write $\dfrac{d}{dx}$, $\dfrac{d}{dy}$, $\dfrac{d}{dz}$, in place of the contragredients, and call τ so altered (τ), then $(\tau) h(\omega)$ will be an invariant of 6 dimensions in the coefficients of ω. This sextinvariant I have little doubt is identical with that obtained by operating upon ω with the commutantive symbol

$$\begin{vmatrix} \left(\dfrac{d}{dx}\right)^2, & \dfrac{d}{dx}\dfrac{d}{dy}, & \left(\dfrac{d}{dy}\right)^2, & \dfrac{d}{dy}\dfrac{d}{dz}, & \left(\dfrac{d}{dz}\right)^2, & \dfrac{d}{dz}\dfrac{d}{dx} \\ \left(\dfrac{d}{dx}\right)^2, & \dfrac{d}{dx}\dfrac{d}{dy}, & \left(\dfrac{d}{dy}\right)^2, & \dfrac{d}{dy}\dfrac{d}{dz}, & \left(\dfrac{d}{dz}\right)^2, & \dfrac{d}{dz}\dfrac{d}{dx} \end{vmatrix}.$$

This, like every other commutant of 2 lines only, is of course capable of being expressed under the form of an ordinary determinant, and the remark is not without interest, as showing how the proposition known with respect to quadratic functions of any number of variables, namely of every such having an invariantive determinant, lends itself to the general case of functions of any even degree of any number of variables which also have always an invariantive determinant attached to them, of which the terms are simple coefficients of such functions. The only peculiarity (if it be one) of quadratic functions in this respect being that they have each but one invariant of such form and no other. In the case before us, if we write

$$\omega = a_1 x^4 + b_2 y^4 + c_3 z^4 + 4a_2 x^3 y + 4a_3 x^3 z + 4b_1 y^3 x + 4b_3 y^3 z + 4c_1 z^3 x + 4c_2 z^3 y$$
$$+ 6dy^2 z^2 + 6ez^2 x^2 + 6fx^2 y^2 + 12lx^2 yz + 12mxy^2 z + 12nxyz^2,$$

the sextinvariant in question becomes representable under the form of the determinant

$$\begin{vmatrix} a_1, & a_2, & f, & l, & e, & a_3 \\ a_2, & f, & b_1, & m, & n, & l \\ f, & b_1, & b_2, & b_3, & d, & m \\ l, & m, & b_3, & d, & c_2, & n \\ e, & n, & d, & c_2, & c_3, & c_1 \\ a_3, & l, & m, & n, & c_1, & e \end{vmatrix}^{*}.$$

* This determinant is identical with the determinant formed by taking the second differential coefficients of the function and arranging in the usual manner the coefficients of the several powers and combinations of powers of the variables treated as if they were independent quantities.

Before quitting the subject of S and T the two invariants of the cubic function of 3 variables, or, as it may be termed, of the cubic curve, it may not be amiss to give the complete table which I have formed corresponding to all the singular cases which can befall such curve, which will be seen below to be eight in number; it is of the highest importance to push forward the advanced posts of geometry, and for this purpose to obtain the same kind of absolute power and authority over, and clear and absolute knowledge of, the properties and affections of cubic forms as have been already attained for forms of the second degree.

Let $$U = ax^3 + 4bx^2y + 4cx^2z + \&c.$$

(1)   When U has one double point $S^3 + T^2 = 0$.

(2)   When U has two double points, that is becomes a conic and right line

$$\frac{dS}{da}\frac{dT}{db} - \frac{dS}{db}\frac{dT}{da} = 0, \&c. \&c.$$

(3)   When U has a cusp $S = 0$, $T = 0$.

(4)   When U has two coincident double points, that is, is a conic and a tangent line thereto, which comprises the two preceding cases in one,

$$\frac{dT}{da} = 0, \quad \frac{dT}{db} = 0, \&c.$$

and also *therefore* $S = 0.$

(5)   When U becomes three right lines forming a triangle

$$\frac{d^2S}{da\,db}\frac{d^2T}{dc\,de} - \frac{d^2T}{da\,db}\frac{d^2S}{dc\,de} = 0, \&c.$$

where a, b, c, e each represent any of the coefficients arbitrarily chosen, whether distinct or identical.

Another, and lower in degree system of equations, may be substituted for the above, obtained by affirming the equality of the ratios between the coefficients of U and the corresponding coefficients of its Hessian.

(6)   When U represents a pencil of three rays meeting in a point

$$\frac{dS}{da} = 0, \quad \frac{dS}{db} = 0, \&c.$$

and also *therefore* $T = 0.$

Also in place of this system may be substituted the system obtained by taking all the coefficients of the Hessian zero.

(7) When U becomes a line, and two other coincident lines,

$$\frac{dS}{da}=0, \qquad \frac{dS}{db}=0, \ \&c.$$

and also

$$\frac{d^2T}{da^2}=0, \qquad \frac{d^2T}{da\,db}=0, \ \&c.$$

I have not ascertained whether this second system necessarily implies the first; I rather think that it does not. In the preceding case also it would be interesting to show the direct algebraical connexion between the system formed by the coefficients of the Hessian and the system consisting of the first derivatives of S.

(8) When U becomes a perfect cube representing three coincident right lines

$$\frac{d^2S}{da^2}=0, \qquad \frac{d^2S}{da\,db}=0, \ \&c.$$

and

$$\frac{d^2T}{da^2}=0, \qquad \frac{d^2T}{da\,db}=0, \ \&c.$$

The first of these systems of equations necessarily implies the equations $\frac{dT}{da}=0, \frac{dT}{db}=0, \&c.$, as is obvious from the equation

$$T=\frac{dS}{da}\frac{d^3H}{dx^3}+\frac{dS}{db}\frac{d^3H}{dx^2dy}+\ \&c.$$

but not necessarily the second and lower system $\frac{d^2T}{da^2}=0, \&c.$ above written.

So if we take

$$u = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$$

when 2 roots are equal

$$s^2 + t^2 = 0,$$

when 2 pairs of roots are equal

$$\frac{ds}{da}\frac{dt}{db}-\frac{ds}{db}\frac{dt}{da}=0, \ \&c.,$$

when 3 roots are equal $s=0, \quad t=0,$

and when all 4 roots are equal

$$\frac{dt}{da}=0, \qquad \frac{dt}{db}=0, \ \&c.$$

Before closing this Section I may make a remark, in reference to the sextic invariant of ω, which admits of being extended to all commutants formed by operating upon the function with a commutantive symbol obtained by writing over one another lines consisting of powers of $\frac{d}{dx}, \frac{d}{dy}$, &c. and

their combinations (to which, in the Third Section, I gave the name of *compound* commutants, a qualification which, for reasons that will hereafter be adduced, I think it advisable to withdraw). The remark I have to make is this, namely that the invariant obtained by operating upon ω with

$$\left. \begin{array}{cccccc} \left(\dfrac{d}{dx}\right)^2, & \dfrac{d}{dx}\dfrac{d}{dy}, & \left(\dfrac{d}{dy}\right)^2, & \dfrac{d}{dy}\dfrac{d}{dz}, & \left(\dfrac{d}{dz}\right)^2, & \dfrac{d}{dz}\dfrac{d}{dx} \\[3ex] \left(\dfrac{d}{dx}\right)^2, & \dfrac{d}{dx}\dfrac{d}{dy}, & \left(\dfrac{d}{dy}\right)^2, & \dfrac{d}{dy}\dfrac{d}{dz}, & \left(\dfrac{d}{dz}\right)^2, & \dfrac{d}{dz}\dfrac{d}{dx} \end{array} \right\},$$

is precisely the same as may be obtained by operating with

$$\left. \begin{array}{cccccc} \dfrac{d}{du}, & \dfrac{d}{dv}, & \dfrac{d}{dw}, & \dfrac{d}{dp}, & \dfrac{d}{dq}, & \dfrac{d}{dr} \\[3ex] \dfrac{d}{du}, & \dfrac{d}{dv}, & \dfrac{d}{dw}, & \dfrac{d}{dp}, & \dfrac{d}{dq}, & \dfrac{d}{dr} \end{array} \right\}$$

upon the concomitant quadratic function to ω obtained by the method of un-ravelment, as in Note (2) of the Appendix [p. 322 above]; and so, in general, every commutant obtained by operating upon a function of any number of variables of the degree $2mp$ with a symbol consisting of $2p$ lines in which the mth powers of $\dfrac{d}{dx}$, $\dfrac{d}{dy}$, &c. and their mth combinations occur, will be identical with the commutant obtained by operating with a symbol also of $2p$ lines, in which only the simple powers occur of $\dfrac{d}{du}$, $\dfrac{d}{dv}$, &c. (where u, v, &c. are cogredient with x^p, $x^{p-1}y$, &c.), upon a function of u, v, &c., formed by the method of unravelment from the given function.

Finally, before quitting the subject of reciprocity, I may state, it follows from the general statement made at the commencement of this Section, that inasmuch as

$$(x\xi + y\eta + z\zeta + \&\text{c.})^2$$

is a universal concomitant form, so also must

$$\left(\dfrac{d}{d\xi}\dfrac{d}{dx} + \dfrac{d}{d\eta}\dfrac{d}{dy} + \dfrac{d}{d\zeta}\dfrac{d}{dz} + \&\text{c.}\right)^2$$

be a universal concomitant symbol of operation; accordingly it is certain that any concomitant in which x, y, z, &c., ξ, η, ζ, &c. enter, operated upon with this symbol, will remain a concomitant: in several cases which I have examined, the effect of this operation will be to produce an evanescent form, but I see no ground for supposing that this is other than an accidental, or at all events for supposing that it is a necessary and universal consequence of the operation. It may also be observed that in the case of as many cogredient sets of variables as variables in each set, as for instance 3 sets

of 3 variables each, the determinant which may be formed by arranging them in regular order, as

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix},$$

is evidently a universal concomitant, and moreover an equivocal concomitant, possessing the property of remaining a concomitant when the variables are respectively but simultaneously exchanged for their contragredients ξ, η, ζ; ξ', η', ζ'; ξ'', η'', ζ''; which shows also that in place of the variables may be written the differential operators

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \frac{d}{dx'}, \frac{d}{dy'}, \frac{d}{dz'}; \frac{d}{dx''}, \frac{d}{dy''}, \frac{d}{dz''}:$$

a remark which leads us to see the exact place in the general theory occupied by Mr Cayley's method of generating covariants given in the concluding paragraph of the First Section [p. 290 above]. I may likewise add, that inasmuch as $(x'\xi + y'\eta + z'\zeta + \&c.)^2$ is a universal concomitant,

$$\left(x' \frac{d}{dx} + y' \frac{d}{dy} + \&c. \right)^r$$

will be so too, by virtue of the general law of interchange, which conducts immediately to the theory of emanation, showing that this last symbol, operating upon any function, furnishes covariants thereunto for any integer value of z.

One additional interesting remark presents itself to be made concerning U, the cubic function of x, y, z, which is, that calling as before T its sextic invariant, and a, $3b$, $3c$, d, &c. the coefficients, the formula

$$\left(\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \xi^2\zeta \frac{d}{dc} + \xi\eta\zeta \frac{d}{dd} + \&c. \right)^2 T$$

will give the polar reciprocal, or, as it has been agreed to term it, the reciprocant of U. I believe the remark of the probability of this being the case originated with myself, but Mr Cayley first verified it by actual calculation, using for that purpose the value of T, given by Mr Salmon in his work *On the Higher Plane Curves*, already frequently alluded to, which is an indispensable manual equally for the objects of the higher special geometry as for the new or universal algebra, being in fact a common ground where the two sciences meet and render mutual aid.

Mr Salmon also observed, that the first evect of T, namely

$$\left(\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \&c. \right) T,$$

was identical in form with what may be termed the first devect of the polar reciprocal, that is, the result of operating upon the polar reciprocal with what U becomes when $\dfrac{d}{d\xi}$, $\dfrac{d}{d\eta}$, $\dfrac{d}{d\zeta}$, are substituted in the stead of x, y, z. And inasmuch as, by Euler's law,

$$\left\{ a\left(\frac{d}{d\xi}\right)^3 + 3b\left(\frac{d}{d\xi}\right)^2\frac{d}{d\eta} + \&c. \right\} \times \left\{ \xi^2\frac{d}{da} + \xi^2\eta\frac{d}{db} + \&c. \right\} T$$

$$= 6\left\{ a\frac{d}{da} + b\frac{d}{db} + \&c. \right\} T = 36T,$$

it follows that T is the second *devect* of the polar reciprocal, or at least identical with it in point of form. But, since the preceding matter was printed, I have discovered in the course of a most instructive and suggestive correspondence with Mr Salmon, the principle upon which these and similar identifications depend, thereby dispensing with the necessity for the excessively tedious labour of verification which, even in the simple example before us, would be found to extend over several pages of work.

The theory in which this principle is involved will be given, along with other very important matter, in the next number of the *Journal*.

Supplementary Observations on the Method of Reciprocity.

It has been observed, that ξ, η, &c. may always be inserted in place of $\dfrac{d}{dx}$, $\dfrac{d}{dy}$, &c., and *vice versâ*, in a concomitant form, without destroying its concomitance. Accordingly, instead of the evector symbol

$$\xi^3\frac{d}{da} + \xi^2\eta\frac{d}{db} + \&c.,$$

we may employ

$$\left(\frac{d}{dx}\right)^3\frac{d}{da} + \left(\frac{d}{dx}\right)^2\frac{d}{dy}\frac{d}{db} + \&c.;$$

and operating with this upon any concomitant, the result will be a concomitant. Hence we see, for example, that if we take the concomitant SH formed by the product of the invariant S and the covariant H,

$$\left\{ \left(\frac{d}{dx}\right)^3\frac{d}{da} + \left(\frac{d}{dx}\right)^2\frac{d}{dy}\frac{d}{db} + \&c. \right\} SH$$

will be a covariant; in fact this will be found to be T, the difference between this and the expression before given for T, namely

$$\left(\frac{d}{dx}\right)^3 H\frac{dS}{da} + \left(\frac{d}{dx}\right)^2\frac{d}{dy}H\frac{dS}{db} + \&c.,$$

being

$$S \times \left\{ \frac{d}{da}\left(\frac{d}{dx}\right)^3 H + \frac{d}{db}\left(\frac{d}{dx}\right)^2\frac{d}{dy}H + \&c. \right\},$$

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which is zero, there being no invariant to $(x, y, z)^3$ of the 3rd degree in a, b, c, &c., as the factor multiplied by S would be were it not evanescent. The same observation may be extended to analogous equations given previously.

I have chiefly, however, made the above observation with a view to making more clear the enunciation of the theorem which I am now about to state, the most important perhaps in its application of any yet brought to light on the subject, but the consequences of which, as I have but quite recently discovered it, must be reserved for a future number of the *Journal*.

Let any function of any number of variables be supposed to have for its coefficients the letters a, b, &c. affected with the ordinary binomial or multinomial coefficients; and let another function be taken identical with the former in all respects, except in the circumstance that all their numerical multipliers are suppressed. Let this function or form be termed the respondent to the primitive: furthermore, by the inverse of any form understand what that form becomes when, in place of x, y, z, &c., ξ, η, ζ, &c.,

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \text{&c.}, \quad \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}, \text{&c.},$$

are respectively substituted (and so for all the systems of the variables), and likewise at the same time similar substitutions are made of $\dfrac{d}{da}, \dfrac{d}{db}, \dfrac{d}{dc}$, &c., in place of a, b, c, &c.; then we have this grand and simple law—*The inverse of any concomitant to a respondent is a concomitant to its primitive.* When the inverse of any concomitant to the respondent is made to operate upon the same concomitant of the primitive, it will be found that the result is a power of the universal concomitant. If the concomitant to the respondent be an invariant thereof, the rule indicates that on merely replacing in the respondent a, b, c, &c. by $\dfrac{d}{da}, \dfrac{d}{db}, \dfrac{d}{dc}$, &c., the result operating on any invariant or other concomitant of the primitive, leaves it still an invariant or other concomitant. For instance, if we take the function

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5,$$

which has three invariants L, M, N, of the degrees 4, 8, 12, respectively: and if we call λ, μ, ν what L, M, N become when, in place of a, b, c, d, e, f respectively, we write

$$\frac{d}{da}, \frac{1}{5}\frac{d}{db}, \frac{1}{10}\frac{d}{dc}, \frac{1}{10}\frac{d}{dd}, \frac{1}{5}\frac{d}{de}, \frac{d}{df},$$

we shall find that

$$\lambda M = L, \quad \mu N = L,$$

and

$$\lambda N = \text{a linear function of } M \text{ and } L^2.$$

Again, if in the case of any function of x, y, z, &c., we take, instead of any other concomitant to the respondent, the respondent itself, its inverse gives the symbol of operation

$$\left(\frac{d}{da}\right)\left(\frac{d}{dx}\right)^3 + \frac{d}{db}\left(\frac{d}{dx}\right)^2\left(\frac{d}{dy}\right) + \&c.,$$

just previously treated of. If again, in the case of a function of x, y, say

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n,$$

we take the inverse of the polar reciprocal of the respondent, we get the operator

$$\frac{d}{da}\left(\frac{d}{d\eta}\right)^n - \frac{d}{db}\left(\frac{d}{d\eta}\right)^{n-1}\frac{d}{d\xi} + \&c.;$$

and replacing $\dfrac{d}{d\eta}$, $\dfrac{d}{d\xi}$ by y, x, we find that

$$y^n\frac{d}{da} - y^{n-1}x\frac{d}{db} + \&c.,$$

operating on any concomitant, leaves it still a concomitant, which is M. Eisenstein's theorem before adverted to, only generalized by the introduction of any concomitant in lieu of the discriminant.

This extraordinary theorem of respondence will be found on reflection to favour the notion of treating the coefficients of a general function as themselves a system of variables, in a manner contragredient to the terms to which they are affixed.

Finally, there is yet another mode of applying the principle of reciprocity, which must be carefully distinguished from any previously stated in these pages.

I have said that in place of the quantitative symbols of one alphabet, as x, y, z, &c., we may always substitute the operation symbols $\dfrac{d}{d\xi}$, $\dfrac{d}{d\eta}$, $\dfrac{d}{d\zeta}$, &c. of the opposite alphabet. But now I say, in place of the quantitative symbols x, y, z, &c. occurring in the concomitant to any form f, may be substituted the quantities (observe, no longer operative symbols but quantities) $\dfrac{dF}{d\xi}$, $\dfrac{dF}{d\eta}$, $\dfrac{dF}{d\zeta}$, &c., F being itself any concomitant to f. Thus, for instance, taking F identical with f, we see that $f\left(\dfrac{df}{d\xi}, \dfrac{df}{d\eta}, \dfrac{df}{d\zeta}, \&c.\right)$ is concomitant to f: or again, if f be a function of x, y only, say $f(x, y)$, taking F the polar reciprocal of f, that is $f(-\eta, \xi)$, we see that $f\left(-\dfrac{df}{dy}, \dfrac{df}{dx}\right)$ will be a

concomitant to f: this concomitant, by the way it may be observed, will always contain f as a factor, because when $f = 0$, $x\dfrac{df}{dx} + y\dfrac{df}{dy} = 0$. Possibly it may be true that, when f is a function of any number of variables x, y, z, &c., and $F(\xi, \eta, \zeta, \&c.)$ its polar reciprocal,

$$f\left(\frac{dF(x, y, z, \&c.)}{dx}, \frac{dF(x, y, z, \&c.)}{dy}, \&c.\right),$$

which is a concomitant to f, contains f as a factor; but I have not had time to see how this is. It is rather singular that Mr Cayley and Professor Borchardt of Berlin have both independently made to me the observation that, when $f(x, y)$ is taken a cubic function of x and y, $f\left(\dfrac{df}{dy}, \dfrac{-df}{dx}\right)$ is equal to the product of f by the first evectant of the discriminant of f. The general consideration of the consequences of this new and important application of the idea of reciprocity must be reserved for a future section.

SECTION V. *Applications and Extension of the Theory of the Plexus.*

If
$$\phi = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

we can obtain, by operating catalectically with x', y' upon

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^2\phi, \quad \left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4\phi,$$

the two concomitants

$$\begin{vmatrix} ax^2 + 2bxy + cy^2, & bx^2 + 2cxy + dy^2 \\ bx^2 + 2cxy + dy^2, & cx^2 + 2dxy + ey^2 \end{vmatrix}, \tag{1}$$

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}, \tag{2}$$

the one in fact being the Hessian, the other the catalecticant of ϕ itself. Again, if
$$\phi = ax^5 + 5bx^4y + 10cx^3y^2 + \ldots + fy^5,$$

by operating catalectically with x', y' upon the second and fourth emanants, as in the last case, we obtain the two covariants

$$\begin{vmatrix} ax^3 + 3bx^2y + 3cxy^2 + dy^3, & bx^3 + 3cx^2y + 3dxy^2 + ey^3 \\ bx^3 + 3cx^2y + 3dxy^2 + ey^3, & cx^3 + 3dx^2y + 3exy^2 + fy^3 \end{vmatrix}, \tag{1}$$

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy \\ bx + cy, & cx + dy, & dx + ey \\ cx + dy, & dx + ey, & ex + fy \end{vmatrix},$$
(2)

which are in fact the Hessian and canonizant respectively of ϕ. So in general, for a function of x, y of the degree 2ι or $2\iota + 1$, we can obtain ι covariantive forms, the first being the Hessian, and the last the catalecticant on the first supposition and the canonizant on the second: calling the index of the function for either case n, the forms appearing in this scale will be of the degree $(r + 1)$ in the constants, and of the degree $(r + 1)(n - 2r)$ in x and y.

It has previously* been intimated that all these determinants admit of a remarkable transformation.

This transformation may be expressed more elegantly by dealing not directly with the covariant forms as above given, but with their polar reciprocants obtained immediately by writing ξ for $-y$ and η for x.

(1) Suppose $\phi = ax^3 + 2bx^2y + 3cxy^2 + dy^3$;

$$\begin{vmatrix} a, & 2b, & c \\ b, & 2c, & d \\ \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix}$$

will be found to be the reciprocant of its Hessian.

(2) Let $\phi = ax^4 + 4bx^3y + \ldots + ey^4$;

the reciprocant of its Hessian will be found to be

$$\begin{vmatrix} a, & 3b, & 3c, & d \\ b, & 3c, & 3d, & e \\ \xi^2, & 2\xi\eta, & \eta^2, & \\ & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix}.$$

(3) Let $\phi = ax^5 + 5bx^4y + \ldots + fy^5$;

the reciprocant of its Hessian will be

$$\begin{vmatrix} a, & 4b, & 6c, & 4d, & e \\ b, & 4c, & 6d, & 4e, & f \\ \xi^2, & 2\xi\eta, & \eta^2, & & \\ & \xi^2, & 2\xi\eta, & \eta^2, & \\ & & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix};$$

[* p. 325 above, note †].

and the reciprocant of its canonizant is

$$\begin{vmatrix} a, & 3b, & 3c, & d \\ b, & 3c, & 3d, & e \\ c, & 3d, & 3e, & f \\ \xi^3, & 3\xi^2\eta, & 3\xi\eta^2, & \eta^3 \end{vmatrix}.$$

The numerical coefficients in this and in the first case are inserted for the sake of uniformity, but it will of course be readily observed that when there is but one line of ξ and η, that the numerical coefficients being the same for each column may be rejected without affecting the form of the result.

So again, if

$$\phi = ax^6 + 6bx^5y + \ldots + gy^6,$$

the reciprocant of the Hessian is

$$\begin{vmatrix} a, & 5b, & 10c, & 10d, & 5e, & f \\ b, & 5c, & 10d, & 10e, & 5f, & g \\ \xi^2, & 2\xi\eta, & \eta^2, & & & \\ & \xi^2, & 2\xi\eta, & \eta^2, & & \\ & & \xi^2, & 2\xi\eta, & \eta^2, & \\ & & & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix},$$

and the reciprocant of the second form in the scale, which comes between the Hessian and the catalecticant, is

$$\begin{vmatrix} a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g \\ \xi^3, & \xi^2\eta, & \xi\eta^2, & \eta^3, & \\ & \xi^3, & \xi^2\eta, & \xi\eta^2, & \eta^3 \end{vmatrix};$$

and so in general. The rule of formation is sufficiently plain not to need formulating in general terms. It is easy to see that all these forms are concomitants to the function from which they are formed; for example, take

$$\phi = ax^6 + 6bx^5y + \ldots + gy^6;$$

then

$$\left(\frac{d}{dx}\right)^2 \phi, \quad \frac{d}{dx}\frac{d}{dy}\phi, \quad \left(\frac{d}{dy}\right)^2 \phi$$

form a plexus.

So likewise if we take $\psi = (x\xi + y\eta)^4$,

$$\frac{d\psi}{d\xi}, \quad \frac{d\psi}{d\eta}$$

form a plexus. But ψ and ϕ are concomitantive, ψ being a universal concomitant. Hence we may combine together these two plexuses, that is

$$\left. \begin{array}{l} ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \\ bx^4 + 4cx^3y + 6dx^2y^2 + 4exy^3 + fy^4 \\ cx^4 + 4dx^3y + 6ex^2y^2 + 4fxy^3 + gy^4 \end{array} \right\},$$

$$\left. \begin{array}{l} \xi^3x^4 + 3\xi^2\eta x^3y + 3\xi\eta^2 x^2y^2 + \eta^3 xy^3 \\ \xi^3x^3y + 3\xi^2\eta x^2y^2 + 3\xi\eta^2 xy^3 + \eta^3 y^4 \end{array} \right\},$$

and, by the principle of the plexus, $x^4, x^3y, x^2y^2, xy^3, y^4$ may be eliminated dialytically, and the resultant will be the determinant last given, which is therefore a contravariant to ϕ.

The manner in which I was led to notice this singular transformation is somewhat remarkable.

In the supplemental part of my essay *On Canonical Forms* [p. 203 above], my method of solution of the problem of throwing the quintic function of two variables under the form $u^5 + v^5 + w^5$, led me to see that u, v, w are the three factors of

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \\ bx+cy, & cx+dy, & dx+ey \\ cx+dy, & dx+ey, & ex+fy \end{vmatrix};$$

the more simple mode of the solution of the same problem, given by me in the *Philosophical Magazine* for the month of November last [p. 266 above], led to

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ y^3, & -xy^2, & x^2y, & -x^3 \end{vmatrix},$$

as the product of the same three factors; whence the identity of the two forms becomes manifest. In the paper last named I gave two proofs, one my own, the other Mr Cayley's, of a like kind of identity for the canonizant of any odd-degreed function of x, y in general. The proof of the identity of the corresponding forms in the much more general proposition above indicated [p. 325 above, footnote †] must be reserved until more pressing and important matters are disposed of. In the footnote referred to I ought to have added, in order to make the sense more clear, that the degree of the catalecticant there referred to in respect of the coefficients would be n.

I regret to think that there are many other typographical errors in the earlier sections; the most unfortunate of these is in the note at page [316], in the values of P and Q belonging to the cubic commutant dodecadic function of x and y, the corrected values of which will be given in my next communication. I ought also to observe, in correction of the remark made in the footnote to page [302], that it follows as a consequence of a recent paper by Dr Hesse in *Crelle's Journal*, that the method given by me in the text applied (according to what I have there termed the 1st process for obtaining an invariant resembling the resultant) to a system of three cubic equations (in which application only the 1st powers of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ enter) produces for that case also, as well as for the cases specified in the note, not a counterfeit resemblance of, but the actual resultant itself.

Returning to the theory of the plexus of which I am about to enunciate a most important extension, I beg to refer my readers to the last paragraph, p. [291], in the last number of the *Journal*, where I have shown how to form, under certain conditions, a determinant by combining together various concomitants and eliminating dialytically one set of the variables, which determinant will be concomitantive to the concomitants out of which it is formed, and of course also therefore to their common original.

Now the extension of this theorem, to which I wish to call attention, is this, that not only such determinant as a whole is a concomitant to such original, but every minor system of determinants that can be formed out of it will form a concomitantive plexus complete within itself to the same original. But, much more generally, it should be observed that there is no occasion to begin with a square determinant; it is sufficient to have a rectangular array of terms formed by taking the several terms of one plexus or of several plexuses combined, provided that they are of the same degree in respect to the variables (or to the selected system of variables, if there be several systems), and forming out of such rectangular array any minor system of determinants at will. Every such system will be a concomitantive plexus. The simple illustrations which follow will make my meaning clear.

Suppose

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 21dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6.$$

I have previously remarked, in the foregoing sections, that a, b, c, d, e, f, g, the coefficients form an invariantive plexus to ϕ; so also we know that the catalecticant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

is an invariant to ϕ. But we are now able to couple together these facts and see the law which is contained between them; for if we take

$$\left(\frac{d}{dx}\right)^\iota \phi, \quad \left(\frac{d}{dx}\right)^{\iota-1} \frac{d}{dy} \phi \cdots \left(\frac{d}{dy}\right)^\iota \phi,$$

ι being any number, as for instance, if we take $\iota = 3$, we shall have as a plexus

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$
$$bx^3 + 3cx^2y + 3dxy^2 + ey^3,$$
$$cx^3 + 3dx^2y + 3exy^2 + fy^3,$$
$$dx^3 + 3ex^2y + 3fxy^2 + gy^3;$$

accordingly not only is the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

an invariant, but also the system obtained by striking out any one line and one column, being what I term the first minors, will be an invariantive plexus, so too will the system of second minors

$$ac - b^2, \; bd - c^2, \; ce - d^2, \; ad - bc, \; ae - bd, \; be - cd, \; \&c.$$

form an invariantive plexus, as well as the last minors, that is, the simple terms a, b, c, d, e, f, g. Again, we might have taken the plexus

$$\left(\frac{d}{dx}\right)^2 \phi, \quad \frac{d}{dx}\frac{d}{dy} \phi, \quad \left(\frac{d}{dy}\right)^2 \phi,$$

which would give the array

$$\begin{array}{ccccc} a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g; \end{array}$$

but the minor systems of determinants herein comprised will be found to be identical with those last considered, with the exception that the highest system, containing a single determinant only, will now be wanting. So in general it will easily be seen that a similar method in general, when ϕ is of 2ι dimensions, will lead to $\iota + 1$ invariantive plexuses comprising the given coefficients grouped together at one extremity of the scale, and the catalecticant alone at the other; and if ϕ is of $2\iota + 1$ dimensions, there will still be $\iota + 1$ such plexuses, commencing with the coefficients as one group and ending with a system of combinations of the $(\iota + 1)$th degree in regard to the coefficients, which system accordingly takes the place of the catalecticant of the former case, which for this case is non-existent.

As a profitable example of the application of this law of synthesis, in its present extended form, let it be required to determine the conditions that a function of x, y of the fifth degree may have three equal roots. In general, let $\phi = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5$, then ϕ has a quadratic and cubic covariant of which I have written at large in my supplemental essay above referred to, being in fact the s and t (that is the quadrinvariant and cubinvariant) in respect to x', y' (x, y being treated as constants) of

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi.$$

Let these covariants respectively be called

$$Ax^2 + 2Bxy + Cy^2 = u,$$

$$\alpha x^3 + 3\beta x^2 y + 3\gamma xy^2 + \delta y^3 = v\,;$$

then

$$\left.\begin{array}{c} Ax + By \\ Bx + Cy \end{array}\right\}$$

forms a plexus, and

$$\left.\begin{array}{c} \alpha x^2 + 2\beta xy + \gamma y^2 \\ \beta x^2 + 2\gamma xy + \delta y^2 \end{array}\right\}$$

will form another.

Now when $a = 0$, $b = 0$, $c = 0$, ϕ will have three equal roots, and

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi$$

becomes

$$6dy \,.\, x'^2 y'^2 + 4\,(dx + ey)\,x'y'^3 + (ex + fy)\,y'^4,$$

of which the quadrinvariant in respect to x', y' is easily seen to be d^2y^2 and the cubinvariant d^3y^3. Accordingly the grouping

$$\begin{array}{c} A,\ B \\ B,\ C \end{array} \text{ becomes } \left.\begin{array}{c} 0,\ 0 \\ 0,\ d^2 \end{array}\right\},$$

and the grouping

$$\begin{array}{c} \alpha,\ \beta,\ \gamma \\ \beta,\ \gamma,\ \delta \end{array} \text{ becomes } \left.\begin{array}{c} 0,\ 0,\ 0 \\ 0,\ 0,\ d^3 \end{array}\right\}.$$

Accordingly, we see that the determinant $\begin{vmatrix} A,\ B \\ B,\ C \end{vmatrix}$ and all the first minors of $\begin{vmatrix} \alpha,\ \beta,\ \gamma \\ \beta,\ \gamma,\ \delta \end{vmatrix}$, that is $\alpha\gamma - \beta^2$, $\beta\delta - \gamma^2$, $\alpha\delta - \beta\gamma$, become zero; but the former single quantity $\begin{vmatrix} A,\ B \\ B,\ C \end{vmatrix}$ being an invariant, and this last system being an invariantive plexus, all the quantities so affirmed to be zero will remain zero, notwithstanding any linear transformations to which ϕ may be subjected; thus then we obtain an immediate proof of the theorem that

when a function of x and y of the fifth degree contains three equal roots the determinant of its quadratic covariant, which in fact is its sole quart-invariant, and the first minors of its cubinvariant will be all separately zero. This theorem may be made still more stringent; for by combining

$$Ax^2 + 2Bxy + Cy^2,$$

$$\alpha x^2 + 2\beta xy + \gamma y^2,$$

$$\beta x^2 + 2\gamma xy + \delta y^2,$$

it becomes manifest that in the case supposed all the first minor determinants of

$$\begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}$$

will be zero, showing in addition to the theorem last enunciated that also

$$A : B : C :: \alpha : \beta : \gamma :: \beta : \gamma : \delta.$$

It is curious and instructive to remark that this last set of equations, stringent as *they appear*, and far more than enough to express a duplex condition, are not sufficient to imply unequivocally the existence of three equal roots, unless we have also $AC - B^2 = 0$; for suppose ϕ to take the form $ax^5 + fy^5$ (b, c, d, e all vanishing); then it will easily be seen that

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0,$$

$$A = 0, \quad B = af, \quad C = 0.^*$$

---

* If we take L, M, N a system of fundamental invariants to ϕ, of which all the other invariants of ϕ are rational integer functions, then $L = \begin{vmatrix} A, & B \\ B, & C \end{vmatrix}$ and the simplest forms for M and N are

$$M = \begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix} \quad \text{and} \quad N = \begin{vmatrix} \alpha, & 2\beta, & \gamma \\ & \alpha, & 2\beta, & \gamma \\ \beta, & 2\gamma, & \delta, \\ & \beta, & 2\gamma, & \delta \end{vmatrix},$$

where L and N are the discriminants of the quadratic and cubic covariants of ϕ respectively, and a linear function of M, L^2 is the discriminant of ϕ itself (L, M, N being of 4, 8, and 12 dimensions respectively in the coefficients of ϕ).

For many purposes of the calculus of forms it is desirable to have the command of cases for which any two out of these three invariants may be made to vanish without the third vanishing; and it will be found that when ϕ is of the form $y^2(cx^3 + fy^3)$, $L = 0$, $M = 0$; when ϕ is of the form $y(bx^4 + fy^4)$, $N = 0$, $L = 0$; and when ϕ is of the form $ax^5 + ey^5$, $M = 0$, $N = 0$; and of course when ϕ is of the form $y^3(dx^2 + fy^2)$, $L = 0$, $M = 0$, $N = 0$; it being obviously true in general, as remarked by Mr Cayley, that when not less than half the roots of a function of two variables are equal, all its invariants must vanish together.

Consequently we shall still have all the first minors of

$$\begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}$$

zero, although there is not even so much as a pair of equal roots in ϕ; $AC - B^2$ however, it will be observed, is not zero in this supposition.

The theory of Hessians, simple or bordered, may be regarded as one among the infinite diversity of applications of the principle of the plexus. Let U, V, W, &c. be any number of concomitants having the common system of variables $x, y \ldots z$. Let χ represent

$$x' \frac{d}{dx} + y' \frac{d}{dy} + \ldots + z' \frac{d}{dz},$$

and take

$$\chi^2 U + \lambda \chi V + \&c. + \mu \chi W = S;$$

then

$$\frac{dS}{dx'}, \frac{dS}{dy'} \ldots \frac{dS}{dz'}$$

forms a plexus; and this, combined with χV, &c. $\ldots \chi W$, enables us to eliminate dialytically x', y', z', $\lambda \ldots \mu$. The result is a Hessian of U, bordered with

$$\frac{dV}{dx}, \frac{dV}{dy} \ldots \frac{dV}{dz}$$

horizontally and vertically, and also with

$$\frac{dW}{dx}, \frac{dW}{dy} \ldots \frac{dW}{dz},$$

$$\&c. \qquad \&c.$$

similarly dispersed; which Hessian, so bordered, is thus seen to be a concomitant to U, $V \ldots W$. The Hessian, as ordinarily bordered with $\xi, \eta \ldots \zeta$, is derived by taking for V the universal concomitant

$$x\xi + y\eta + \ldots + z\zeta,$$

and for W (if there be a double border)

$$x\xi' + y\eta' + \ldots + z\zeta',$$

and so forth.

If V be taken identical with U, the resulting form, consisting of U bordered with $\frac{dU}{dx}, \frac{dU}{dy} \ldots \frac{dU}{dz}$, has been shown* in my paper "On certain general Properties of Homogeneous Functions," in this *Journal*, to be equal to the product of the simple Hessian of U and of U itself multiplied by a

[* p. 173 above.]

numerical factor. The theory of the bordered Hessian may be profitably extended by taking

$$S = \chi^{2r} U + \lambda \chi^r V + \ldots + \mu \chi^r W,$$

and combining with $\chi^r V \ldots \chi^r W$ the plexus obtained by operating upon S with the rth powers and products of $\frac{d}{dx}, \frac{d}{dy} \ldots \frac{d}{dz}$, and eliminating dialytically the rth powers and products of $x', y' \ldots z'$. Thus if

$$U = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \text{ and } V = (x\xi + y\eta)^2,$$

we obtain, by taking $S = \chi^4 U + \lambda \chi^2 V$, and proceeding as indicated in the preceding,

$$\begin{vmatrix} a, & b, & c, & \xi^2 \\ b, & c, & d, & \xi\eta \\ c, & d, & e, & \eta^2 \\ \xi^2, & \xi\eta, & \eta^2, & \end{vmatrix}$$

as a concomitant to U. So again, if

$$U = ax^5 + 5bx^4y + \ldots + fy^5,$$

we find

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy, & \xi^2 \\ bx+cy, & cx+dy, & dx+ey, & \xi\eta \\ cx+dy, & dx+ey, & ex+fy, & \eta^2 \\ \xi^2, & \xi\eta, & \eta^2, & \end{vmatrix}$$

a concomitant to U.

These extensions of the ordinary theory of Hessians will be found to be of considerable practical importance in the treatment of forms, for which reason they are here introduced.

SECTION VI. *On the Partial Differential Equations to Concomitants, Orthogonal and Plagiogonal Invariants, &c.*

In the 7th note of the Appendix to the three preceding sections* I alluded to the partial differential equations by which every invariant may be defined.

This method may also be extended to concomitants generally. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr Cayley who communicated to me the equations which define the invariants of functions of

[* p. 326 above.]

two variables*. The method by which I obtain these equations and prove their sufficiency is my own, but I believe has been adopted by Mr Cayley in a memoir about to appear in *Crelle's Journal*. I have also recently been informed of a paper about to appear in *Liouville's Journal* from the pen of M. Eisenstein, where it appears the same idea and mode of treatment have been made use of. Mr Cayley's communication to me was made in the early part of December last, and my method (the result of a remark made long before) of obtaining these and the more general equations, and of demonstrating their sufficiency, imparted a few weeks subsequently— I believe between January and February of the present year.

The method which I employ, in fact, springs from the very conception of what an invariant means, and does but throw this conception into a concise analytical form.

Suppose, to fix the ideas,

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2} n (n-1) cx^{n-2}y^2 + \ldots + ly^n,$$

and let $I(a, b, c \ldots l)$ be any invariant to ϕ.

Now suppose x to become $x + ey$, but y to remain unchanged; the modulus of the transformation, $\begin{vmatrix} 1, & e \\ 0, & 1 \end{vmatrix}$, being unity, I cannot alter in consequence of this substitution; but the effect of this substitution is to convert ϕ into the form

$$\alpha x^n + n\beta x^{n-1}y + \tfrac{1}{2} n (n-1) \gamma x^{n-2}y^2 + \ldots + \lambda y^n,$$

where $\qquad \alpha = a, \quad \beta = b + ae, \quad \gamma = c + 2be + ae^2, \text{ \&c. \&c.}$

$$\lambda = l + \ldots + nbe^{n-1} + ae^n.$$

Consequently, if we make

$$\Delta b = ae, \quad \Delta c = 2be + ae^2, \text{ \&c. \&c.,}$$

we have by Taylor's theorem, observing that $\Delta a = 0$,

$$\Delta I = \left(\Delta b \frac{d}{db} + \Delta c \frac{d}{dc} + \text{\&c.}\right) I + \frac{1}{1.2}\left(\Delta b \frac{d}{db} + \Delta c \frac{d}{dc} + \text{\&c.}\right)^2 I$$

$$+ \frac{1}{1.2.3}\left(\Delta b \frac{d}{db} + \text{\&c.}\right)^3 I + \text{\&c.} = 0;$$

---

* It is extremely desirable to know whether M. Aronhold's equations are the same in form as those here subjoined. It is difficult to imagine what else they can be in substance. Should these pages meet the eye of that distinguished mathematician he will confer a great obligation on the author and be rendering a service to the theory by communicating with him on the subject: and I take this opportunity of adding that I shall feel grateful for the communication of any ideas or suggestions relating to this new Calculus from any quarter and in any of the ordinary mediums of language—French, Italian, Latin or German, provided that it be in the Latin character.

and this being true for all the values of e, every separate coefficient of e in ΔI must be zero: hence we obtain n different equations by equating to zero the coefficients of e, $e^2 \dots e^n$ respectively. The first of these equations will be

$$\left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \&c. \right) \phi = 0,$$

and it is obvious that this will imply all the rest; for, when e is taken indefinitely small, $I(a, b, c \dots)$ does not alter (when this equation is satisfied) by changing a, b, $c \dots$ into a', b', $c' \dots$; consequently $I(a', b', c', \&c.)$ will not alter, when in place of a', b', c' we write a'', b'', c'', &c., obtained from a', b', c', &c., by the same law as a', b', c', &c., from a, b, c, &c.

Thus we may go on giving an indefinite number of increments, ey to x, without changing the value of I. Consequently, if the equation above written be satisfied, *à priori* all the rest must be so too. But there is not any difficulty in showing the same thing by a direct method*.

For we have

$$\left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \&c. \right) I = 0,$$

an identical equation. Hence

$$\left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \&c. \right) \left\{ \left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \&c. \right) I \right\} = 0;$$

hence

$$\left\{ \left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \&c. \right) \left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \&c. \right) \right\} I$$

$$+ \left\{ a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \&c. \right\}^2 I = 0,$$

that is

$$\left\{ 2 \left(a \frac{d}{dc} + 3b \frac{d}{dd} + 6c \frac{d}{de} + \&c. \right) + \left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \&c. \right)^2 \right\} I = 0;$$

repeating the application of the symbolic operator

$$\left(a \frac{d}{db} + 2b \frac{d}{dc} + \&c. \right),$$

---

* The method above given has the advantage however of being immediately applicable to every species of concomitant, and we learn from it that concomitance, whether absolute or conditional, is sufficiently determined when affirmed to exist for *infinitesimal* variations; it cannot exist for infinitesimal variations without, by *necessary implication*, existing for finite variations also; a most important consideration this in conducing to a true idea of the nature of invariance and the other kinds of concomitance, and in cutting off all superfluous matter from the statement of the conditions by which they are defined.

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we obtain

$$
\left.
\begin{aligned}
& 1 \cdot 2 \cdot 3 \left\{ a\,\frac{d}{dd} + 4b\,\frac{d}{de} + 10c\,\frac{d}{df} + \&c. \right\} \\
& + 1 \cdot 2 \left\{ a\,\frac{d}{db} + 2b\,\frac{d}{dc} + \&c. \right\} \left\{ a\,\frac{d}{dc} + 3b\,\frac{d}{dd} + \&c. \right\} \\
& + \left(a\,\frac{d}{db} + 2b\,\frac{d}{dc} + 3c\,\frac{d}{dd} + \&c. \right)^{3}
\end{aligned}
\right\} I = 0,
$$

and so on; the numerical multipliers of the terms of the several series within the parentheses forming the regular succession of figurate numbers

$$1, \quad 2, \quad 3, \ \&c.$$
$$1, \quad 3, \quad 6, \ \&c.$$
$$1, \quad 4, \quad 10, \ \&c.$$

It is easy to see that these equations correspond to the results of making the coefficients of the successive powers of e equal to zero.

I may remark, that the first instance as far as I know on record of this, (as some may regard it rather bold) but in point of fact perfectly safe and legitimate method of differentiating conjointly operator and operand, occurs in a paper by myself in this *Journal*, Feb. 1851, "On certain General Properties of Homogeneous Functions" [p. 165 above]; where I have applied it in operating with

$$
\left\{ (x_1 - a_1 e)\,\frac{d}{da_1} + (x_2 - a_2 e)\,\frac{d}{da_2} + \&c. \right\}
$$

upon

$$
\left\{ (x_1 - a_1 e)\,\frac{d}{da_1} + (x_2 - a_2 e)\,\frac{d}{da_2} + \&c. \right\}^{r} \omega,
$$

which, as I have there noticed, gives the result

$$
\left\{ (x_1 - a_1 e)\,\frac{d}{da_1} + \&c. \right\}^{r+1} \omega
$$

$$
- re \left\{ (x_1 - a_1 e)\,\frac{d}{da_1} + \&c. \right\}^{r} \omega.
$$

The equation $\left(a\,\dfrac{d}{db} + 2b\,\dfrac{d}{dc} + \&c. \right) I = 0$ is evidently not enough to define I as an invariant; it merely serves to show that I does not alter when in place of x we write $x + ey$, but this is true for any function of the differences of the roots of the form multiplied by a suitable power of a, namely that power which is just sufficient to cause the product to become integer. But if we now, for convenience, write

$$
\phi = ax^{n} + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^{2} + \dots
$$
$$
+ \tfrac{1}{2}n(n-1)c'x^{2}y^{n-2} + nb'xy^{n-1} + a'y^{n},
$$

and form the similar equation from the other side, namely

$$\left(a'\frac{d}{db'} + 2b'\frac{d}{dc'} + 3c'\frac{d}{dd'} + \&c.\right)I = 0,$$

these two equations together will suffice to define any invariant, as I shall proceed to show—these are the two equations alluded to brought under my notice by Mr Cayley. If they coexist, it follows from the method by which I have deduced them that x may be changed into $x + ey$, or y into $y + fx$, without I being altered, e and f having any values whatever: and it is obvious that these substitutions may be performed, not merely alternatively but successively, because the equations between the coefficients are identical equations, and depend only on the form of I.

Let now x become $x + ey$, and then y become $y + fx$; the result of these substitutions is to convert

$$x \text{ into } x + efx + ey,$$

and y into $fx + y.$

Finally, let x become $x + gy$; then x is converted into $(1 + ef)(x + gy) + ey$, and y into $y + f(x + gy)$,

that is x becomes $(1 + ef)x + (eg + efg)y$,

and y becomes $fx + (1 + fg)y.$

The modulus of substitution it is evident, *à priori*, always remains unity, and nothing would be gained by pushing the substitutions any further, as it is clear that we may satisfy the equations

$$1 + ef = p, \quad e + g + efg = q,$$
$$f = p', \quad 1 + fg = q',$$

for all values of p, q, p', q', which satisfy the equation

$$pq' - p'q = 1,$$

and for none other except such values; hence I remains unaltered for any unit-modular linear transformation of x, y, and is therefore an invariant by definition.

If ϕ be taken a function of three variables, x, y, z, and be thrown under the form

$$az^n + (a_1 x + b_1 y)z^{n-1} + (a_2 x^2 + 2b_2 xy + c_2 y^2)z^{n-2} + \&c.,$$

and I be any invariant of ϕ, by supposing x to become $x + ey$, and giving b_1, b_2, c_2, &c., the corresponding variations, and taking e indefinitely small, we obtain

$$\left\{a_1\frac{d}{db_1} + \left(a_2\frac{d}{db_2} + 2b_2\frac{d}{dc_2}\right) + \left(a_3\frac{d}{db_3} + 2b_3\frac{d}{dc_3} + 3c_3\frac{d}{dd_3}\right) + \&c.\right\}I = 0,$$

$$\left\{b_1\frac{d}{da_1} + \left(c_2\frac{d}{db_2} + 2b_2\frac{d}{da_2}\right) + \&c. \&c.\right\}I = 0:$$

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and in like manner, by arranging ϕ according to the powers of y and of x, we obtain two other pairs of equations: it is clear, however, that three equations (it would seem any three out of the six) would suffice and imply the other three. The method of demonstration will be the same as in the instance of two variables: First, it can be shown by the method of successive accretions, that I remaining invariable when x receives an indefinitely small increment ϵy, or y an indefinitely small increment ϵz, or z an indefinitely small increment ϵx, it will also remain invariable when these increments are taken of any finite magnitude. Secondly, by eight successive transformations, admissible by virtue of the preceding conclusion, x, y, z may be changed into any linear functions of x, y, z, consistent with the modulus of transformation being unity. And in general for a function of m variables, m partial differential equations similarly constructed (but not however arbitrarily selected) will be necessary and sufficient to determine any invariant: and it is clear that all the general properties of invariants must be contained in and be capable of being educed out of such equations.

The same method enables us also to establish the partial differential equations for any covariant, or indeed any concomitant whatever.

Thus let

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \ldots + nb'xy^{n-1} + a'y^n = 0,$$

and let $K(a, b, c, \&c.; x, y, x', y', \&c.; \xi, \eta, \&c.)$ represent any concomitant, x, y; x', y' being cogredient, and ξ, η, &c. contragredient systems; when x, y become $x+ey$, y, any such system x', y' becomes $x'+ey'$, y'; and any such system as ξ, η becomes ξ, $\eta - e\xi$; and taking e indefinitely small, the second coefficients a, b, c, &c. become a, $b+ae$, $c+2be$, &c. as before; hence the equation to the concomitant becomes

$$\left\{ a\frac{d}{db} + 2b\frac{d}{dc} + \ldots - y\frac{d}{dx} - y'\frac{d}{dx'} + \ldots + \xi\frac{d}{d\eta} - \&c. \right\} = 0^*;$$

and in like manner, by changing y into $y+ex$, results the corresponding equation

$$\left\{ a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \ldots - x\frac{d}{dy} - x'\frac{d}{dy'} + \ldots + \eta\frac{d}{d\xi} - \&c. \right\} K = 0.$$

These two equations define in a perfectly general manner every concomitant (with any given number of cogredient and contragredient systems) to the form ϕ; and the due number of pairs of similarly constituted equations will serve to define the concomitant to a function of any given number of variables†.

---

* For we have

$$K(a, b+ae, c+2be, \&c.; x, y, \&c.; \xi, \eta, \&c.)$$

$$= K(a, b, c, \&c.; x, x+ey, \&c.; \xi, \eta - e\xi, \&c.; \&c.).$$

† *Vide* Note (10) [p. 361 below].

In like manner we may proceed to form the equations corresponding to what may be termed *conditional* concomitants, whether *orthogonal* or *plagiogonal*. The concomitants previously considered may be termed absolute, the linear transformations admissible being independent of any but the one general relation, imposed merely for the purpose of convenience, namely of their modulus being made unity. An orthogonal concomitant is a form which remains invariable, not for arbitrary unit-modular, but for orthogonal transformation, that is for linear substitutions of x, $y \ldots z$, which leave unchanged $x^2 + y^2 + \ldots + z^2$: in like manner, a plagiogonal concomitant may be defined of a form which remains invariable for all linear substitutions of x, $y \ldots z$, which leave unaltered any given quadratic function of x, $y \ldots z$. Thus, let it be required to express the condition of $Q\,(a, b, c \ldots x, y ; \xi, \eta)$, being an orthogonal concomitant to the form

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n.$$

Let x become $x + ey$, e being indefinitely small, then y must become $y - ex$, and the variations of $a, b \ldots b', a'$ will be the sum of the variations produced by taking separately $x + ey$ for x and $y - ex$ for y. Hence the one sole condition for Q being of the required form becomes

$$\left\{ \begin{aligned} &\left(a\,\frac{d}{db} + 2b\,\frac{d}{dc} + \ldots - y\,\frac{d}{dx} + \xi\,\frac{d}{d\eta} \right) \\ &- \left(a'\,\frac{d}{db'} + 2b'\,\frac{d}{dc'} + \ldots - x\,\frac{d}{dy} + \eta\,\frac{d}{d\xi} \right) \end{aligned} \right\} Q = 0,$$

or, as it may be written, $\theta Q - \omega Q = 0$, where $\theta Q = 0$, $\omega Q = 0$ are the two equations expressing the conditions of Q, being an unconditional or absolute concomitant; and so in general if ϕ be a function of m variables, we may obtain $\frac{1}{2}m(m-1)$ equations of the form $L - M = 0$ for the concomitant, of which however $(m-1)$ only will be independent.

Supposing, again, the substitutions to which x, y are subject to be conditioned by $lx^2 + 2mxy + ny^2$ remaining unalterable, or which is a more convenient and only in appearance less general supposition by $x^2 + 2mxy + y^2$ remaining unalterable, the general type of an infinitesimal system of substitutions will be rendered by supposing x, y to become $(1 + me)x + ey$, $-ex + (1 - me)y$, respectively, for then $x^2 + 2mxy + y^2$ becomes

$$(1 - m^2e^2)\,x^2 + \{2m + (2m - 2m^2)\,e^2\}\,xy + (1 - m^2e^2)\,y^2,$$

which differs from $x^2 + 2mxy + y^2$ only by quantities of the second order of smallness which may be neglected, and ξ and η will therefore become $(1 - me)\,\xi - e\eta$, $-ex + (1 + me)\,y$, respectively: then, as to the coefficients of ϕ, in addition to the variations which they undergo when m is zero, there will be the variations consequent upon x assuming the increment mex, and y

the increment $-mey$: but by making x become $x + mex$, a, b, c, &c., b', a' assume respectively the variations

$$n \cdot mea, \ (n-1) \, meb, \ \ldots \ meb', \ 0, \text{ respectively};$$

and by making y become $y - mey$, the corresponding variations become

$$0, \ - meb, \ \ldots - (n-1) \, meb', \ -n \cdot mea', \text{ respectively.}$$

Hence the equation becomes

$$\theta Q - \omega Q + m \left(\lambda Q - \mu Q \right) = 0,$$

where θ and ω have the same signification as before, and where λ denotes

$$na \frac{d}{da} + (n-1) \, b \frac{d}{db} + \ldots + b' \frac{d}{db'} + x \frac{d}{dx} - \xi \frac{d}{d\xi},$$

and μ denotes

$$b \frac{d}{db} + 2c \frac{d}{dc} + \ldots\ldots\ldots + na' \frac{d}{da'} - y \frac{d}{dy} + \eta \frac{d}{d\eta}.$$

If there be several systems of x, y or of ξ, η, or of both, the only difference in the equation of condition will consist in putting

$$\Sigma \left(y \frac{d}{dx} \right), \quad \Sigma \left(x \frac{d}{dy} \right), \quad \Sigma \left(x \frac{d}{dx} \right), \quad \Sigma \left(y \frac{d}{dy} \right),$$

$$\Sigma \left(\eta \frac{d}{d\xi} \right), \quad \Sigma \left(\xi \frac{d}{d\eta} \right), \quad \Sigma \left(\xi \frac{d}{d\xi} \right), \quad \Sigma \left(\eta \frac{d}{d\eta} \right),$$

instead of the single quantities included within the sign of definite summation.

Fearing to encroach too much on the limited space of the *Journal*, I must conclude for the present with showing how to integrate the general equation to the orthogonal invariant of ϕ, the general function of x, y.

Beginning with $\phi = ax^2 + 2bxy + cy^2$, the equation becomes

$$\left\{ - 2b \frac{d}{da} + (a-c) \frac{d}{db} + 2b \frac{d}{dc} + y \frac{d}{dx} - x \frac{d}{dy} \right\} Q = 0.$$

Write now

$$da = - 2bd\theta, \qquad dx = yd\theta,$$
$$db = (a-c) \, d\theta, \quad dy = - xd\theta,$$
$$dc = + 2bd\theta;$$

we have then

$$\lambda da + \mu db + \nu dc = d\theta \left\{ \mu a + 2 (\nu - \lambda) \, b - \mu c \right\}.$$

Let

$$\mu = \kappa \lambda, \quad 2 (\nu - \lambda) = \kappa \mu, \quad - \mu = \kappa \nu;$$

then

$$d \log (\lambda a + \mu b + \nu c) = \kappa d\theta;$$

or

$$\lambda a + \mu b + \nu c = b e^{\kappa \theta}.$$

To find κ we have the determinant

$$\begin{vmatrix} \kappa, & -1, & 0 \\ 2, & \kappa, & -2 \\ 0, & 1, & \kappa \end{vmatrix} = 0,$$

that is,

$$\kappa^3 + 4\kappa = 0,$$

and calling the three roots of this equation $\kappa_1, \kappa_2, \kappa_3$, we have

$$\kappa_1 = 0, \quad \kappa_2 = 2\iota, \quad \kappa_3 = -2\iota ;$$

accordingly we may put

$$\kappa = 0, \qquad \lambda = 1, \quad \mu = 0, \qquad \nu = 1,$$

or

$$\kappa = 2\iota, \qquad \lambda = 1, \quad \mu = 2\iota, \qquad \nu = -1,$$

or

$$\kappa = -2\iota, \qquad \lambda = 1, \quad \mu = -2\iota, \quad \nu = -1.$$

Again,

$$pdx + qdy = (py - qx)\, d\theta ;$$

and putting $-q = ep$, $p = eq$, so that $px + qy = Ee^{e\theta}$,

$$e^2 = -1, \quad e_1 = \iota, \quad e_2 = -\iota ;$$

and we may put

$$e = \iota, \qquad p = 1, \quad q = -\iota,$$

or

$$e = -\iota, \qquad p = 1, \quad q = +\iota.$$

Consequently the complete integral of the given partial differential equation is found by writing

$$a + c = l, \qquad\qquad x - \iota y = Ee^{\theta},$$

$$a + 2\iota b - c = l'e^{2\iota\theta}, \qquad x + \iota y = E'e^{-\iota\theta},$$

$$a - 2\iota b - c = l''e^{-2\iota\theta}.$$

By means of these five equations, after eliminating θ, we may obtain four independent equations between a, b, c; x, y. Suppose

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0, \quad Q_4 = 0;$$

then $Q = F(Q_1, Q_2, Q_3, Q_4)$ is the complete integral required.

Pursuing precisely the same method for the general case, it will be found that, calling the degree of the given function n when n is even, the equation in κ to be solved will be

$$\kappa (\kappa^2 + 4)(\kappa^2 + 9) \ldots (\kappa^2 + n^2) = 0;$$

and when n is odd (say $2m + 1$), the equation in κ to solve will be

$$(\kappa + 1)(\kappa^2 + 9) \ldots (\kappa^2 + n^2) = 0;$$

and performing the necessary reductions, and calling the roots of the equation, arranged in order of magnitude, $\kappa_1\iota$, $\kappa_2\iota$ ... $\kappa_n\iota$, respectively, it will be found that the equations containing the integral become

$$
\left.
\begin{aligned}
L_1 &= l_1 e^{\kappa_1\iota\theta} \\
L_2 &= l_2 e^{\kappa_2\iota\theta} \\
L_3 &= l_3 e^{\kappa_3\iota\theta} \\
&\cdots\cdots\cdots \\
L_{n+1} &= l_{n+1} e^{\kappa_{n+1}\iota\theta}
\end{aligned}
\right\}
\quad
\left.
\begin{aligned}
x - \iota y &= E e^{\iota\theta} \\
x + \iota y &= E' e^{-\iota\theta}
\end{aligned}
\right\},
$$

where l_1, l_2 ... l_{n+1}; E, E' are arbitrary constants, and where L_1, L_2 ... L_{n+1} are the values assumed by the 1st, 2nd ... $(n+1)$th coefficients of the given function ϕ, or

$$
ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n,
$$

when it is transformed by writing $x + \iota y$ in place of x, and $y + \iota x$ in place of y. ι is of course employed in the foregoing according to the usual notation to represent $\sqrt{(-1)}$. The same method applies to the general theory of plagiogonal concomitants, where the linear substitutions are supposed such as to leave $lx^2 + 2mxy + ny^2$ unaltered in form, and the equations in θ which contain the integral present themselves under a similar aspect. But a more full discussion of these interesting integrals must be reserved until the ensuing number of the *Journal*.

NOTES IN APPENDIX.

(9) The scale of covariants to a function of (x, y) obtained by the method of unravelment [on p. 297 above], may be otherwise deduced in a form more closely analogous to that of the corresponding theorems for the corresponding invariantive scale [on p. 295 above], by a method which has the advantage of exhibiting the scale equally well for the case of functions of the degree $4\iota + 2$ or $4\iota + 4$, the only difference being that in the latter case the coefficients of the odd powers of λ will be found all to vanish, so that the degrees of the covariants will rise by steps of 4 instead of by steps of 2, just conversely to what happens in the invariantive scale; whereas in the invariantive scale alluded to the forms containing odd powers of λ vanish when the degree of the function is of the form $4\iota + 2$, but do not vanish when it is of the form 4ι. This method in the form here subjoined is a slight modification of one suggested to me by my friend Mr Cayley.

Let F be the given function of x, y of the degree $2n$; take the systems x', y'; x_1, y_1 cogredient with one another and with x, y. Then form the concomitant

$$
K = \left(x' \frac{d}{dx} + y' \frac{d}{dy} \right)^n F + \lambda (x'y - y'x)^{n-1} (x'y_1 - y'x_1)(xy_1 - yx_1).
$$

Then (by what may be termed the Divellent method, which has been previously applied by me in the *Philosophical Magazine* for Nov. 1851) calling θ_0, θ_1, $\theta_2 \dots \theta_n$, the coefficients of

$$x'^n, \ x'^{n-1}, \ y', \ \dots y'^n \text{ in } K,$$

we shall have

$$\theta_0 = A_0 x^n + B_0 x^{n-1} y + \dots + L_0 y^n,$$

$$\theta_1 = A_1 x^n + B_1 x^{n-1} y + \dots + L_1 y^n,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\theta_n = A_n x^n + B_n x^{n-1} y + \dots + L_n y^n,$$

the coefficients being functions of the coefficients of f and of quadratic combinations of x_1, y_1, affected with the multiplier λ; and the determinant

$$\begin{vmatrix} A_0, & B_0 \dots L_0 \\ A_1, & B_1 \dots L_1 \\ \dots, \dots\dots\dots \\ A_n, & B_n \dots L_n \end{vmatrix}$$

will give a function of λ in which the coefficients of the several powers of λ will be all zero or covariants of F.

The actual form of this determinant is not here given for want of space and time, but will be exhibited hereafter. Precisely an analogous method applies to obtain the scale to $(x, y, z)^4$ given in Note (2) [p. 322 above]. Calling $F = (x, y, z)^4$, let the systems x', y', z'; x_1, y_1, z_1, be taken cogredient with one another and with x, y, z. Then, using R to express the determinant

$$\begin{vmatrix} x', & y', & z' \\ x, & y, & z \\ x_1, & y_1, & z_1 \end{vmatrix},$$

and making

$$K = \left(x' \frac{d}{dx} + y' \frac{d}{dy} + z' \frac{d}{dz} \right)^2 F + \lambda R,$$

and proceeding as above by the divellent method, we obtain the scale required.

(10) [p. 356 above.] It is obvious that these defining equations ought to give the means of discovering and verifying all the properties of concomitants; but it is very difficult to see how in the present state of analysis many of the general theorems that have been stated, readily admit of being deduced from them.

The comparatively simple but eminently important theory of the evector symbol does however admit of a very pretty verification by aid of these equations. Thus, suppose θ any concomitant; suppose a contravariant to a function F of x, y, say

$$ax^n + nbx^{n-1}y + \dots + nb'xy^{n-1} + a'y^n.$$

Then θ must satisfy the two equations

$$\left(L + \xi \frac{d}{d\eta}\right)\theta = 0, \quad \left(L' + \eta \frac{d}{d\xi}\right)\theta = 0,$$

where

$$L = a\frac{d}{db} + 2b\frac{d}{dc} + \ldots + nb'\frac{d}{da'},$$

$$L' = a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \ldots + nb\frac{d}{da}.$$

Now let $\phi = \chi(\theta)$ where

$$\chi = \xi^n\frac{d}{da} + \xi^{n-1}\eta\frac{d}{db} + \xi^{n-2}\eta^2\frac{d}{dc} + \ldots + \eta^n\frac{d}{da'};$$

then

$$L(\chi\theta) = \chi(L\theta) - (\chi L)\theta$$

$$= \chi(L\theta) - \left(\xi^n\frac{d}{db} + 2\xi^{n-1}\eta\frac{d}{dc} + \ldots + n\xi\eta^{n-1}\frac{d}{da'}\right)\theta,$$

$$\xi\frac{d}{d\eta}(\chi\theta) = \chi\left(\xi\frac{d}{d\eta}\theta\right) + \left(\xi\frac{d}{d\eta}\chi\right)\theta$$

$$= \chi\left(\xi\frac{d}{d\eta}\theta\right) + \left(\xi^n\frac{d}{db} + 2\xi^{n-1}\eta\frac{d}{dc} + \ldots + n\xi\eta^{n-1}\frac{d}{da'}\right)\theta.$$

Hence

$$\left(L + \xi\frac{d}{d\eta}\right)\chi(\theta) = \chi\left\{\left(L + \xi\frac{d}{d\eta}\right)\theta\right\} = \chi(0) = 0.$$

Similarly

$$\left(L' + \eta\frac{d}{d\xi}\right)\chi(\theta) = 0.$$

Hence if θ is an integral of the two conditioning equations, so also is $\chi(\theta)$. In like manner, if θ be a covariant or any other kind of concomitant of F, it may be proved that its evectant $\chi(\theta)$ is the same.

(11) [p. 331 above.] Very much akin with the supposed equations is the following most remarkable equation, which can be proved to exist. Let ϕ be a function of x and y of the 5th degree. Let P and Q be the quadratic and cubic covariants of ϕ. P is of two dimensions in the coefficients and also in the variables, and Q of three dimensions in both; they are in fact the s and t (in respect to x' and y') of $\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4\phi$. Then, giving P and Q proper numerical factors, it will be found that

$$H_2\phi + PH\phi + Q\phi = 0.$$

I believe that a similar equation connects any function of x and y above the 3rd degree with its first and second Hessians. The proof will be given in a subsequent Section, where also I shall give a complete proof, which occurred to me immediately after sending the preceding note to the press, of the complete Theory of the Respondent by means of the general equations of concomitance.

P.S. Since the preceding was in type, I have ascertained the existence and sufficiency of a general method for forming the polar reciprocal and probably also the discriminant to functions of any degree of three variables by an explicit process of permutation and differentiation. In particular I am enabled to give the actual rule for constructing the polar reciprocal and the discriminant curves of the 4th and 5th degrees. So far as regards the polar reciprocal of curves of the 4th degree M. Hesse has already given a method of obtaining it, but mine is entirely unlike to this, and rests upon certain extremely simple and universal principles of the calculus of forms. The only thing necessary to be done in order to carry on the process to curves of the 6th or higher degrees, is to ascertain the relation of the discriminants of functions of two variables of those respective degrees to such of the fundamental invariants as are of an inferior order to the discriminant.

The theory applies equally well to surfaces and to functions of any number of variables, and may, I believe, without any serious difficulty be extended so as to reduce to an explicit process the general problem of effecting the elimination between functions of any degree and of any number of variables. The method above adverted to will appear in a subsequent Section.