46.

OBSERVATIONS ON A NEW THEORY OF MULTIPLICITY.

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IN the Postscript to my paper in the last number of the *Magazine*, I mis-stated, or to speak more correctly, I understated the law of Evection applicable to functions having any given amount of distributive multiplicity. The law may be stated more perfectly, and at the same time more concisely, as follows. Every point represented by the coordinates α_1, $\beta_1 \dots \gamma_1$, for which the multiplicity is m_1, will give rise in *every evectant* * of the discriminant of the function to a factor $(\alpha_1 x + \beta_1 y + \dots + \gamma_1 z)^{m_1 n}$, n being supposed to be the degree of the function. Hence if there be r such points, for which the several multiplicities are m_1, $m_2 \dots m_r$, every evectant must contain $(m_1 + m_2 + \dots + m_r)\, n$ linear factors; and as the ιth evectant is of the degree ιn, it follows that all the evectants below the $(m_1 + m_2 + \dots + m_r)$th evectant must vanish completely, and this Evectant itself be contained as a factor in all above it†. When a function of only two variables is in question, there is no difficulty in understanding what property of the function it is which is indicated by the allegation of the existence of multiplicities m_1, $m_2 \dots m_r$;

* Frequent use being made in what follows of the word Evectant, I repeat that the evectant of any expression connected with the coefficients of a given function (supposed to be expressed in the more usual manner with letters for the coefficients affected with the proper binomial or polynomial numerical multipliers) means the result of operating upon such expressions with a symbol formed from the given function by suppressing all the binomial or polynomial numerical parts of the coefficients to be suppressed, and writing in place of the literal parts of the coefficients a, b, c, &c. the symbols of differentiation $\frac{d}{da}$, $\frac{d}{db}$, $\frac{d}{dc}$, &c.; in all that follows it is the successive evectants of the discriminant alone which come under consideration. I need hardly repeat, that the discriminant of a function is the result of the process of elimination (clear from extraneous factors) performed between the partial differential quotients of the function in respect to the several variables which it contains, or to speak more accurately, is the characteristic of their coevanescibility.

† The constitution of the quotients obtained by dividing all the other evectants of the discriminant by the first non-evanescent one, presents many remarkable features which remain yet to be fully studied out, and promise a wide extension of the existing theory.

as already remarked, this simply means that there are r distinct groups of equal roots, such groups containing $1+m_1$, $1+m_2 \ldots 1+m_r$ roots respectively. So for curves and higher loci, the total distributive multiplicity is the sum of the multiplicities at the several multiple points. But the true theory of the higher degrees of multiplicity separately considered at any point remains yet to be elaborated, and will be found to involve the consideration of the theory of elimination from a point of view under which it has never hitherto been contemplated.

Confining our attention for the present to curves, we have a clear notion of the multiplicity 1: this is what exists at an ordinary double point. As well known, its analytical character may be expressed by saying that the function of x, y, z, which characterizes the curve, is capable, when proper linear transformations are made, of being expanded under the form of a series descending according to the powers of z, such that the constant coefficient of the highest power of z, and the linear function of x, y, which is the coefficient of the next descending power of z, may both disappear. Again, when the multiplicity is 2, the third coefficient, which is a quadratic function of x and y, will become a perfect square. This is the case of a cusp, which, as I have said, is the precise analogue to that of three equal roots for a function of two variables. Before proceeding to consider what it is which constitutes a multiplicity 3 for a curve, it will be well to pause for a moment to fix the geometrical characters of the ordinary double point and the cusp.

If we agree to understand by a first polar to a curve the curve of one degree lower which passes through all the points in which the curve is met by tangents drawn from an arbitrary point taken anywhere in its own plane, we readily perceive that at an ordinary double point all the infinite number of first polars which can be drawn to the curve will intersect one another at the double point. Again, at a cusp all these polars will not only all intersect, they will moreover all touch one another at the cusp. Now we may proceed to inquire as to the meaning of a multiplicity of the third degree, which, strange to say, I believe has never yet been distinctly assigned by geometricians.

This is not the case of a so-called triple point, that is a point where three branches of the curve intersect. Supposing $x=0$, $y=0$, to represent such a point, the characteristic of the curve must be reducible to the form

$$(gx^3+hx^2y+kxy^2+ly^3)\,z^{n-3}+\&c.,$$

which, as is well known, involves the existence of four conditions. This, however, would not in itself be at all conclusive against the multiplicity at a triple point being only of the third degree; for it can readily be shown that there may exist singular points of any degree of *singularity* (as measured by the number of conditions necessary to be satisfied in order that such singularity may come into existence), but for which the multiplicity may be as low as we please ; as, for instance, if at a double point (which is not a cusp) there be a point of inflexion on one branch or on both, or a point of undulation, or any other singularity whatever, still provided there be no cusps, the multiplicity will stick at the first degree and never exceed it; for only the discriminant itself will vanish on these suppositions, but no evectant of the discriminant. The reason, on the contrary, why a so-called triple point must be said to have a multiplicity of the degree 4, and not merely of the degree 3, springs from the fact that the 1st, 2nd and 3rd evectants of the discriminant all vanish at such a point.

It is clear, then, that there ought to exist a species of multiplicity for which the 1st and 2nd evectants vanish, but not the 3rd. In fact, as at a double point the first polars all merely intersect, but at a cusp have all a contact with one another of the first degree, so we ought to expect that there should exist a species of multiple point such that all the first polars should have with each other a contact of the second degree (or if we like so to say, the same curvature) at that point. When the curve has a triple point, all its first polars will have that point upon them as a double point; and it is not at the first glance, easy *à priori* to say what is the nature of the contact between two curves which intersect at a point which is a double point to each of them: we know upon settled analytical principles, that when one curve having a double point is crossed there by another curve not having a double point, that the two must be said to have with one another, a contact of the 1st degree; and we now learn from our theory of evection, that if each have a double point at the meeting-point, the degree of the contact must from principles of analogy be considered to be of the 3rd degree*. Now, then, we come to the question of deciding definitely what is a multiple point for which the degree of multiplicity is 3. It is, adopting either test, whether of first polar contact or of evection, a cusp situated or having its *nidus*, so to say, at a point of inflexion. In other words, $x = 0$, $y = 0$ will be a point whose multiplicity is intermediate between that of the cusp and that of a so-called triple point, when the characteristic of the curve admits of being written under the form

$$z^{n-2}x^2 + z^{n-3}(gx^3 + hx^2y + ixy^2) + z^{n-4} \text{ \&c.};$$

or in other words, when over and above the vanishing of the constant and linear coefficients, and the quadratic coefficient being a perfect square, as in the case of an ordinary cusp, this square has a factor in common with the next (the cubic) coefficient; or again, in other words, a curve has a point

* This may easily be verified by direct analytical means; as also the more general proposition, that two curves meeting at a point where there are m branches of the one and n branches of the other, must be considered to have mn coincident points in common, that is, if we like so to express it, to have a contact of the degree $mn - 1$.

for which the multiplicity is 3 when its characteristic function admits of being expanded according to the powers of one of the variables, in such a manner that the first coefficient and the second (the linear) coefficient vanish, and that the discriminant of the third and the resultant of the third and fourth are both at the same time zero. This being the case, it may be shown that the first polars will all have with each other a contact of the second degree; and moreover, that all the evectants of the discriminant will have as a common factor a linear function of the variables, raised to a power whose index is three times that of the characteristic function. As, then, there is but one kind of ordinary double point, and but one kind of point with multiplicity 2, so there is one, and only one, kind of point with a multiplicity 3. A cusp is a peculiar double point; a flex-cusp (as for the moment I call the point last above discussed) is a peculiar cusp. This law of unambiguity, however, appears to stop at the third degree. A so-called triple point (which ought in fact to be called a *quintuple* point) is a point for which the multiplicity, as shown above, is of the fourth degree; but it is not the only point of that degree of multiplicity. Without assuming to have exhausted every possible supposition upon which such a degree of multiplicity may be brought into existence, it will be sufficient to take as an example a curve whose characteristic is capable of assuming the form

$$z^{n-2}x^2 + z^{n-3}(gx^3 + hx^2y) + z^{n-4}(kx^4 + lx^3y + mx^2y^2 + nxy^3) + z^{n-5}\,\&c.$$

It may readily be demonstrated that the first polars of this curve have all with one another at the point x, y a contact of a degree exceeding the 2nd, that is of at least the 3rd degree (and, I believe, in general not higher). Now the point x, y is evidently not a triple-branched point, but a cusp with three additional degrees of singularity; so that we have evidence of the existence of a point whose degree of singularity is 5, and whose multiplicity is at least 4, but which is in no sense a modified triple point. It is probably true (but to demonstrate this requires a further advance to be made than has yet been realized in the theory of the constitution of discriminants) that a cusp may be so modified by the *nidus* at which it is posited, as, without ever passing into a triple point, to be capable of furnishing any amount of multiplicity whatever, curiously in this contrasting with an ordinary double point, no amount whatever of extraordinary singularity imparted to which, or so to speak, to its *nidus*, can ever heighten its multiplicity so as to make it surpass the first degree without first converting it into a cusp. I may illustrate the nature of a flex-cusp by what happens to a curve of the third degree. When it breaks up into a conic and a right line, there are two ordinary double points; for the existence of these double points, as for the existence of a cusp, two conditions are required. When, however, the right line and conic touch one another (a *casus omissus* this in the works of the special geometers), the characters of the cusp and the point of inflexion are combined at the point of contact; the multiplicity is of the third degree, and the singularity also of a degree not exceeding this; three conditions only being necessary to be satisfied in order that a given cubic may degenerate into such a form; and it will be found that the discriminant and the first and second evectants thereof vanish for this case, and that the third evectant of the discriminant will be a perfect 9th power; whereas in order that the cubic may have a so-called triple point, that is may degenerate into a trident of diverging rays, four conditions must be satisfied, and it will be found that when this is the case, the first, second, and third evectants of the discriminant will all vanish, and the fourth will be a perfect 12th power of a linear function of the variables. I may mention, by the way, at this place, that the law of a discriminant and the successive evectants up to the mth inclusive, all vanishing, may be expressed otherwise (not in *identical*, but in *equivalent* or *equipollent* terms), by saying that the discriminant and all its derivatives of a degree not exceeding the mth will all vanish—understanding by a derivative of the discriminant any function obtained from the discriminant by differentiating it any specified number of times with respect to the constants of the function to which it belongs, the same constants being repeated or not indifferently*. And very surprising it must be allowed to be, stated as a bare analytical fact, that $(m+1)$ conditions imposed upon the coefficients of a function of any number of variables and of any degree should suffice to make the inordinately greater number of functions which swarm among the derivatives of the mth and inferior degrees of the discriminant each and all simultaneously vanish.

Without pushing these observations too far for the patience of the general reader, it may be remarked by way of setting foot with our new theory upon the almost unvisited region of the singularities of surfaces, that by the light of analogy we may proceed with a safe and firm step as far as multiplicity of the third degree inclusive.

The function characteristic of the surface being supposed to be expressed in terms of the four variables x, y, z, t, and expanded according to descending powers of t, then when x, y, z is an ordinary double point of the first degree of multiplicity, the constant and the linear coefficient disappear; when the point has a multiplicity 2, the discriminant of the quadratic coefficient will be zero, that is this coefficient will be expressible by means of due linear transformations under the form of x^2+y^2; and when the multiplicity is to be of the degree 3, the cubic coefficient will, at the same time that the quadratic coefficient is put under the form x^2+y^2, itself (for the same system of x and y) assume the form of a cubic function of x, y, z, in which the highest power of z, that is z^3, will not appear; or in other words (restoring to x, y, z their

* Or, to speak more simply, the discriminant and its successive *differentials* up to the mth exclusive must all vanish simultaneously.

generality), not only will the first derivatives of the quadratic function be nullifiable simultaneously with each other, but likewise at the same time with the cubic function itself. These three cases will be for surfaces, the analogues so far, but only so far as regards the degree of the multiplicity, to the double point, cusp, and flex-cusp of curves*. The analogue to the so-called triple point of the curves will be a point whose degree of singularity, depending upon the vanishing of the six constants in the third coefficient (which is a quadratic function of x, y, z) at the same time as the three constants in the linear factor, would seem to be but 6 more than for a double point, that is in all $1 + 6$ or 7, but whose multiplicity, as inferred from the nature of the contact of its first polars, which will be of the 7th order, would appear to be 8 (a seeming incongruity which I am not at present in a condition to explain)†; so that there will apparently be 4 steps of multiplicity to interpolate between this case and the case analogous (*sub modo*) to the flex-cusp, last considered. Whether these intervening degrees correspond to singularities of an unambiguous kind, no one is at present in a condition to offer an opinion. I will conclude with a remark, the result of my experience in this kind of inquiry as far as I have yet gone in it, namely that it would be most erroneous to regard it as a branch of isolated and merely curious or fantastic speculation. Every singularity in a locus corresponds to the imposition of certain conditions upon the form of its characteristic; by aid of the theory of evection we are able to connect the existence of these conditions with certain consequences happening to the form of the discriminant, and thereby it becomes possible, upon known principles of analysis, to infer particulars relating to the constitution of the discriminant itself in its absolutely general form, very much upon the same principle as when the values of a function for particular values of its variable or variables are known, the general form of the function thereby itself, to some corresponding extent, becomes known. Thus, for instance, I have by the theory of evection in its most simple application, been led to a representation of the discriminant

* At an ordinary conical point of a surface for which the multiplicity is 1, every section of the surface is a curve with a double point. When the multiplicity is 2, the cone of contact becomes a pair of planes, through the intersection of which any other plane that can be drawn cuts the surface in a section having an ordinary cusp of multiplicity 2, but which themselves cut the surface in sections, having so-called triple points, so that for these two principal sections (which is rather surprising) the multiplicity suddenly jumps up from 2 to 4. All other things remaining unaltered when the multiplicity of the conical point is 3, the cusp belonging to any section of the surface drawn through any intersection of the two tangent planes passes from an ordinary cusp to a flex-cusp.

† So, too, at a so-called quadruple point in a curve, the degree of the contact of the 1st polars is 8, and therefore the multiplicity of the curve at such point is 9; but the number of constants which vanish for this case (namely all those of the cubic coefficient in x, y) over and above what vanish for the case of a so-called triple point is only 4, which is a unit less than the difference between the measures of the multiplicities at the respective points; and this difference continues to increase as we pass on to so-called quintuple and higher multiple points in the curves.

of a function of two variables under a form very different and very much more complete and fecund in consequences than has ever been supposed, or than I had myself previously imagined, to be possible.

According to the opinion expressed by an analyst of the French school, of pre-eminent force and sagacity, it is through this theory of multiplicity, here for the first time indicated, that we may hope to be able to bridge over for the purposes of the highest transcendental analysis, the immense chasm which at present separates our knowledge of the intimate constitution of functions of two from that of three, or any greater number of variables.

It is, as I take pleasure in repeating, to a hint from Mr Cayley*, who habitually discourses pearls and rubies, that I am indebted for the precious and pregnant observation on the form assumed by the first discriminantal evectant of a binary function with a pair of equal roots, out of which, combined with some antecedent reflections of my own, this new theory of multiplicity has taken its rise. The idea of the process of evection, and the discovery of its fundamental property of generating what, in my calculus of forms (*Cambridge and Dublin Mathematical Journal*), I have called *contravariants*, is due to my friend M. Hermite. The polar reciprocals of curves and other loci are contravariants and, as I have recently succeeded in showing, for curves at least, evectants, but of course not discriminantal evectants; and I am already able to give the actual explicit rule for the formation of the polar reciprocal of curves as high as the 5th degree, which with a little labour and consideration can be carried on to the 6th, and in fact to curves of any degree n when once we are acquainted with any mode of determining all such independent invariants of a function of two variables as are of dimensions not exceeding $2(n-1)$ in respect of the coefficients.

By the special geometers (by whom I mean those who, unvisited by a higher inspiration, continue to regard and to cultivate geometry as the science of mere sensible space) this problem has only been accomplished, and that but recently, for curves whose degrees do not exceed the 4th. Mr Salmon has made the happy and brilliant (and by the calculus of forms instantaneously demonstrable) discovery, communicated to me in the course of a most instructive and suggestive correspondence, that *a certain readily ascertainable*

* Mr Cayley's theorem stood thus:—If

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n$$

have two equal roots, and ϖ be its discriminant, then will

$$\left\{y^n \frac{d}{da} - y^{n-1}x \frac{d}{db} \&c. \pm x^n \frac{d}{da'}\right\} \varpi$$

be a perfect nth power. It will easily be seen that this theorem is convertible into a theorem of evection by interchanging in the result x and y with y and $-x$.

*evectant of every discriminant of any function whatever is an exact power of its polar reciprocal**.

I believe that it may be shown, that, with the sole exception of odd-degreed functions of two variables, the *polar reciprocal itself* (as distinguished from a power thereof) of every function is an evectant, not (of course) of the discriminant, but of some determinable inferior invariant.

P.S. The terms pluri-simultaneous and pluri-simultaneity, used or suggested by me in my last paper in the *Magazine*, may be advantageously replaced by the more euphonious and regularly formed words consimultaneous, consimultaneity. Multiplicity and all its attributes and consequences are included as particular cases in the general conception and theory of consimultaneity, that is of consimultaneous equations, or, which is the same thing, of consimulevanescent functions.

* Namely, for a function of degree n, and variability (that is, having a number of variables) p, the $(n-1)^{p-1}$th evect of the discriminant is the $(n-1)$th power of the polar reciprocal.