49.

ON A SIMPLE GEOMETRICAL PROBLEM ILLUSTRATING A CONJECTURED PRINCIPLE IN THE THEORY OF GEOMETRICAL METHOD.

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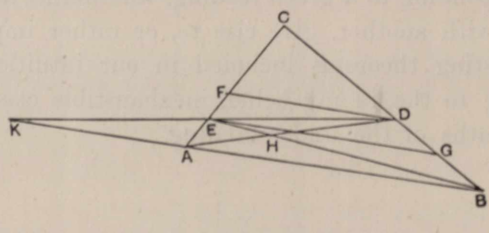
THE following theorem deserves attention as illustrating a principle of geometrical method which will be presently adverted to. It is curious, also, from the fact of its solution being by no means so obvious and self-evident as one would expect from the extreme simplicity of its enunciation. It appeared, and for the first time, it is believed, at the University of Cambridge about a twelvemonth back, where it excited considerable attention among some of the mathematicians of the place. The proposition, as originally presented, was merely to prove that if ABC be a triangle, and if AD and BE drawn bisecting the angles at A and B and meeting the opposite sides in D and E be equal, then the triangle must be isosceles. It is particularly



noticeable that all the geometrical demonstrations yet given of this theorem are indirect. Thus the first and simplest (communicated to me by a promising young geometrician, Mr B. L. Smith of Jesus College, Cambridge), was the following:—Assume one of the angles at DAB to be greater than the corresponding angle EBA; it can easily be shown that, upon this supposition, D will be higher up from AB than E; so that if DF and EG be drawn parallel to AB, DF will be above EG; it is then easily shown that $DF = AF$, $EG = BG$, and consequently DF and AF are each respectively less than EG

and BG; and also DFA, which is the supplement of twice DAB, will be less than EGB, which is the supplement of twice FBA; from which it is readily inferred, by an easy corollary to a proposition of Euclid, that DA will be less than FB, whereas it should be equal to it; so that neither of the half angles at the base can be greater than the other, and the triangle is proved to be isosceles. Another and independent demonstration by the writer of this article is less simple, but has the advantage of lending itself at once to a considerable generalization of the theorem as proposed. Assuming, as above, that DAB is greater than EBA, it is easily seen that DE produced will cut BA at K on the side of it: also if AD and BE intersect in H, it is readily demonstrable, by a suitably constructed apparatus of similar triangles, that

$$AH : BH :: CE : CD.$$

But as HBA is less than HAB, AH is less than BH, and therefore CE is less than CD, and therefore CED is greater than CDE; that is to say, CAB less K is greater than CBA plus K, and therefore DAB less K is greater than EBA, that is ADE is greater than ABE, and therefore the perpendicular from A upon DE is greater than that from E on AB, which is easily proved to be absurd. Hence, as before, the triangle is proved to be isosceles. This proof, it is obvious, remains good for *all cases* in which EB and DA, drawn on either side of the base, divide the angles at the base proportionally, provided that these lines remain equal, and make positive or negative angles with the base not *less* than one-half of the respective corresponding angles which the sides of the triangle are supposed to make with it. The analytical solution of the question, as might be expected, extends the result still further. To obtain this, let

$$BAC = n \,.\, BAD, \qquad ABC = n \,.\, ABE,$$

n for the present being any numerical quantity, positive or negative; calling $BAC = 2n\alpha$, $ABC = 2n\beta$, we readily obtain, by comparison of the equal dividing lines with the base of the triangle,

$$\frac{\sin(2n\alpha + 2\beta)}{\sin 2n\alpha} = \frac{\sin(2n\beta + 2\alpha)}{\sin 2n\beta},$$

or

$$\frac{\sin(2n\alpha + 2\beta)}{\sin(2n\beta + 2\alpha)} = \frac{\sin 2n\alpha}{\sin 2n\beta};$$

and by an obvious reduction,

$$\frac{\tan(n-1)(\alpha-\beta)}{\tan n(\alpha-\beta)} = \frac{\tan(n+1)(\alpha+\beta)}{\tan n(\alpha+\beta)}.$$

When this equation is put under an integer form, it is of course satisfied by making $\alpha = \beta$; on any other supposition than $\alpha = \beta$ it evidently cannot be satisfied by admissible values of the angles for any value of n between

$+1$ and $+\infty$; for on that supposition, since $(\alpha-\beta)$ and $(\alpha+\beta)$ are each less than $\frac{180}{2n}$, the first side of the equation will be necessarily a proper fraction and positive; but the second side, either a positive improper fraction if $(n+1)(\alpha+\beta)$ be less, and a negative proper or a negative improper fraction if $(n+1)(\alpha+\beta)$ be greater than a right angle.

If n be negative, let it equal $-\nu$, then

$$\frac{\tan(\nu+1)(\alpha-\beta)}{\tan\nu(\alpha-\beta)}=\frac{\tan(\nu-1)(\alpha+\beta)}{\tan\nu(\alpha+\beta)};$$

and for the same reason as before, if ν lies between ∞ and 1, this equation cannot be satisfied. Hence the theorem is proved to be true for all values of n, except between $+1$ and -1. For these values it ceases to be true; in fact, for such values for any given values of $(\alpha-\beta)$ there will be always, as it may be easily proved, one or more values of $(\alpha+\beta)$; thus if $n=\frac{1}{2}$, the equation becomes

$$\frac{\tan 3\left(\dfrac{\alpha+\beta}{2}\right)}{\tan\dfrac{\alpha+\beta}{2}}=-1;$$

and if $n=-\frac{1}{2}$,

$$\frac{\tan 3\left(\dfrac{\alpha-\beta}{2}\right)}{\tan\dfrac{\alpha-\beta}{2}}=-1,$$

showing that $\alpha+\beta=90$ and $\alpha-\beta=\pm 90$ in these respective cases will afford a solution over and above the solution $\alpha=\beta$, which is easily verified geometrically*. It would be an interesting inquiry (for those who have leisure for such investigations) to determine for any given value of n between $+1$ and -1 the superior and inferior limits to the number of admissible values of $\alpha+\beta$ corresponding to any given value of $\alpha-\beta$†.

My reader will now be prepared to see why it is that all the geometrical demonstrations given of this theorem, even in the simplest case of all, namely when $n=2$, are indirect, I believe I may venture to say *necessarily* indirect. It is because the truth of the theorem depends on the necessary non-existence of real roots (between prescribed limits) of the analytical equation expressing the conditions of the question; and I believe that it may be safely taken as an axiom in geometrical method, that whenever this is the case no other

* In the first of these cases, if the base of the triangle is supposed given, the locus of the vertex is a right line and a circle; in the second case, a right line and an equilateral hyperbola.

† When $\pm n$ lies between $\frac{1}{2\iota-1}$ and $\frac{1}{2\iota+1}$ (ι being any positive integer), it is easily seen that the superior limit must be at least as great as ι.

form of proof than that of the *reductio ad absurdum* is possible in the nature of things. If this principle is erroneous, it must admit of an easy refutation in particular instances.

As an example, I throw out (not a challenge, but) an invitation to discover a direct proof, if such exist, of the following geometrical theorem, as simple a one as it is perhaps possible to imagine:—" To prove that if from the middle of a circular arc two chords be drawn, and the remoter segments of these chords cut off by the line joining the end of the arc be equal, the nearer segments will also be equal." The analytical proof depends upon the fact of the equation $x^2 + ax = b^2$ (where a is the given length of each segment, and b the length of the chord of half the given arc) having only one admissible root; and if the principle assumed or presumed to be true be valid, no other form of pure geometrical demonstration than the *reductio ad absurdum* should be applicable in this case. For the converse case, where the nearer segments are given equal, the reducing equation is $a(a + x) = b^2$, indicating nothing to the contrary of the possibility of there being a direct solution, which accordingly is easily shown to exist. The indirect form of demonstration, it may be mentioned, is sometimes liable to be introduced in a manner to escape notice. As, for instance, if it should be taken for granted in the course of an argument, that one triangle upon the same base and the same side of it as another triangle, and having the same vertical angle, must have its vertex lying on the same arc; this would seem to be *immediately* true by virtue of the well-known theorem, that angles in the same circular segment are equal, but in reality can only be *inferred* from it indirectly by showing the impossibility of its lying outside or inside the arc in question. To go one step further, I believe it to be the case, that granted to be true all those fundamental propositions in geometry which are presupposed in the principles upon which the language of analytical geometry is constructed, then that the *reductio ad absurdum* not only is of necessity to be employed, but moreover in propositions of an affirmative character never need be employed, except when as above explained the analytical demonstration is founded on the impossibility or inadmissibility of certain roots due to the degree of the equation implied in the conditions of the question. If this surmise turn out to be correct, we are furnished with *a universal criterion for determining when the use of the indirect method of geometrical proof should be considered valid and admissible and when not**.

* If report may be believed, intellects capable of extending the bounds of the planetary system and lighting up new regions of the universe with the torch of analysis, have been baffled by the difficulties of the elementary problem stated at the outset of this paper, in consequence, it is to be presumed, of seeking a form of geometrical demonstration of which the question from its nature does not admit. If this be so, no better evidence could be desired to evince the importance of such a criterion as that suggested in the text.