

NOTE ON THE CALCULUS OF FORMS.

[See pp. 363 and 411.]

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 62—64.]

ACCIDENTAL causes have prevented me from composing the additional sections on the Calculus of Forms, which I had destined for the present Number of this *Journal*. In the meanwhile the subject has not remained stationary. Among the principal recent advances may be mentioned the following.

1. The discovery of Combinants; that is to say, of concomitants to systems of functions remaining invariable, not only when combinations of the variables are substituted for the variables, but also when combinations of the functions are substituted for the functions; and as a remarkable first-fruit of this new theory of double invariability, the representation of the Resultant of any three quadratic functions under the form of the square of a certain combinative sextic invariant added to another combinant which is itself a biquadratic function of 10 cubic invariants. When the three quadratic functions are derived from the same cubic function, this expression merges in M. Aronhold's for the discriminant of the cubic. The theory of combinants naturally leads to the theory of invariability for non-linear substitutions, and I have already made a successful advance in this new direction.

2. The unexpected and surprising discovery of a quadratic covariant to any homogeneous function in x, y of the n th degree, containing $(n-1)$ variables cogredient with $x^{n-2}, x^{n-3}y \dots y^{n-2}$ and possessing the property of indicating *the number of real and imaginary roots* in the given function. This covariant, on substituting for the $(n-1)$ variables the combinations of the powers of x, y with which they are cogredient, becomes the Hessian of the given function*.

* This covariant furnishes, if we please, functions symmetrical in respect to the two ends of an equation for determining the number of its real and imaginary roots. The ordinary Sturmian functions, it is well known, have not this symmetry. As another example of the successful application of the new methods to subjects which have been long before the mathematical world and supposed to be exhausted, I may notice that I obtain without an effort, by their aid, a much more simple, practical, and complete solution of the question of the simultaneous transformation of two quadratic functions, or the orthogonal transformation of one such function, than any previously given, even by the great masters Cauchy and Jacobi, who have treated this question.

3. The demonstration due to M. Hermite of a law of reciprocity connecting the degree or degrees of any function or system of functions with the order or orders of the invariants belonging to the system. The theorem itself was first propounded by me about a twelvemonth back, and communicated to Messrs Cayley, Polignac, and Hermite, as serving to connect together certain phenomena which had presented themselves to me in the theory: unfortunately it appeared to contradict another law too hastily assumed by myself and others as probably true, and I consequently laid aside the consideration of this great law of reciprocity. To M. Hermite, therefore, belongs the honour of reviving and establishing,—to myself whatever lower degree of credit may attach to suggesting and originating,—this theorem of numerical reciprocity, destined probably to become the corner-stone of the first part of our new calculus; that part, I mean, which relates to the generation and affinities of forms*.

4. I may notice that the Calculus of Forms may now with correctness be termed the Calculus of Invariants, by virtue of the important observation that every concomitant of a given form or system of forms may be regarded as an invariant of the given system and of an absolute form or system of absolute forms combined with the given form or system. As regards that particular branch of the theory of invariants which relates to resultants, or, in other words, to the doctrine of elimination, I may here state the theorem alluded to in a preceding Number of the *Journal*, to wit that if R be the resultant of a system of n homogeneous functions of n variables, written out in their complete and most general form (so that by definition $R=0$ is the condition that the equations got by making the n given functions zero, shall be simultaneously satisfiable by one system of ratios), then the condition that these equations may be satisfied by ι distinct systems of ratios between the n variables is $\delta^\iota R=0$, the variation δ being taken in respect to every constant entering into each of the n equations.

* This theorem of numerical reciprocity promises to play as great a part in the Theory of Forms as Legendre's celebrated theorem of reciprocity in that of Numbers. Another demonstration of it, which leaves nothing to be desired for beauty and simplicity, has been since discovered by Mr Cayley, which ultimately rests upon that simple law (essentially although not on the face of it a law of reciprocity) given by Euler, which affirms that the number of modes in which a number admits of being partitioned is the same whether the condition imposed upon the mode of partitionment be that no part shall exceed a given number, or that the number of parts constituting any one partition shall not exceed the same number.