58.

ON THE CONDITIONS NECESSARY AND SUFFICIENT TO BE SATISFIED IN ORDER THAT A FUNCTION OF ANY NUMBER OF VARIABLES MAY BE LINEARLY EQUIVALENT TO A FUNCTION OF ANY LESS NUMBER OF VARIABLES.

[*Philosophical Magazine,* v. (1853), pp. 119—126.]

In the *Cambridge and Dublin Mathematical Journal* for November 1850*, I defined an order as signifying any linear function of a given set of variables, and spoke of a general function of n variables as losing r orders when the relation between its coefficients is such that it is capable of being expressed as a function of $(n-r)$ orders only. It will be highly convenient to preserve the same nomenclature for the purposes of the present investigation.

Dr Otto Hesse, in a long memoir in *Crelle's Journal*, the contents of which have been described to me†, but which I have not yet been able to procure, has given a rule for determining the analytical conditions for the loss of one order. I propose to give a more simple and comprehensive scheme of conditions than Professor Hesse appears to have discovered, applicable not to this case only, but to that of the loss of any number whatever of orders, and shall moreover show in what relation the substituted orders stand to the given variables.

Dr Hesse's rule had been previously stated by me in the 4th section of my Calculus of Forms (*Cambridge and Dublin Mathematical Journal*, May 1852‡) as applicable to the case of a general function of the 3rd degree

[* p. 171 above.]

† A distinguished mathematical friend in Paris communicated to me with great admiration Professor Hesse's result overnight. I ventured to affirm that, to one conversant with the calculus of forms, the problem could offer no manner of difficulty. An hour's quiet reflection in bed the following morning, or morning after, sufficed to disclose to me the true principle of the solution. [Cf. Noether, *Math. Annal.* L. (1898) p. 138. Ed.]

‡ *Vide* Vol. vii. p. 187 [p. 335 above]. "When U represents a pencil of three rays meeting in a point, $\frac{dS}{da}=0$, $\frac{dS}{db}=0$, &c., and also therefore $T=0$" (S and T being the two Aronholdian invariants of U, and a, b, c, &c. the coefficients of U); "also in place of this system may be substituted the system obtained *by taking all the coefficients of the Hessian zero.*"

of three variables becoming the representative of three right lines diverging from the same point, which is the case of a cubic function of three variables becoming a function of two linear functions of these variables, that is to say, losing one order: this, perhaps, might have been noticed in the Professor's memoir. I gave also another rule for the same case; but the true fundamental scheme of conditions about to be set forth will be seen to embrace as mere corollaries all such and such-like rules, which in fact supply more or less arbitrary combinations of the conditions, rather than the naked conditions themselves in their simple form and absolute totality.

I shall call the function to be dealt with U, and shall consider U to be a *homogeneous** rational function of m dimensions in respect of $x_1, x_2 \dots x_n$, and shall inquire what are the conditions which must obtain when U is capable of being expressed as a function of only $(n-r)$ orders, say $l_1, l_2 \dots l_{n-r}$, each of which is of course a homogeneous linear function of the given n variables.

Let the term derivative of U be understood to mean any result obtained by differentiating U any number of times with respect to one or more of the variables $x_1, x_2 \dots x_n$. The first derivatives will be of $(m-1)$ dimensions, the second derivatives of $(m-2)$ dimensions, and so on; and finally, the $(m-1)$th derivatives will be homogeneous linear functions of $x_1, x_2 \dots x_n$. Suppose U to be expressible as a function of $l_1, l_2 \dots l_{n-r}$. It is immediately obvious that the derivatives from the 1st to the $(m-1)$th inclusive will be all expressible as homogeneous functions of $l_1, l_2 \dots l_{n-r}$, and vanish when these vanish. But this statement is in substance pleonastic; for by means of Euler's well-known law, any derivative of U, say K, may be expressed (to a numerical factor *près*) under the form of

$$x_1 \frac{dK}{dx_1} + x_2 \frac{dK}{dx_2} + \dots + x_n \frac{dK}{dx_n},$$

and consequently, whenever the linear derivatives of U vanish, all the upper derivatives of U, including U itself, must vanish at the same time. The number of these linear derivatives, say ν, will be the number of terms in a homogeneous function of n variables of $(m-1)$ dimensions, that is to say,

$$\frac{n(n-1) \dots (n-m+2)}{1 \cdot 2 \dots (m-1)}.$$

Again, if all the ν linear derivatives vanish when the $(n-r)$ equations $l_1 = 0$, $l_2 = 0 \dots l_{n-r} = 0$ are satisfied, r being greater than zero, this can only happen by virtue of these ν derivatives being linear functions of $(n-r)$

* It is a common error to regard homogeneity of expression as merely a means for satisfying the desire for symmetry; the ground of its application and utility in analysis lies, in fact, much deeper; it is essentially a *method* and a *power*.

of them. Now, conversely, I shall prove, that if it be true that all the linear derivatives of U are linear functions $(n-r)$ of them, then U may be expressed as a function of these $(n-r)$; and this rule, as will be immediately made apparent, will give the necessary and sufficient conditions for the loss of r orders in the most simple and complete form by which they admit of being expressed. For the proof of the rule, only one additional remark has to be made in addition to that already made, of the vanishing of the linear derivatives necessarily implying the simultaneous evanescence of all the other derivatives; this additional remark being, that if the derivatives of any class, linear or otherwise, *quâ* one set of variables, become all zero, the derivatives of the same class, *quâ* any other set of variables linear functions of the first set and the same in number, will also become zero, for they are evidently expressible as linear functions of the first set.

Now let $d_1, d_2 \ldots d_{n-r}$ be any $(n-r)$ linear derivatives of U, of which all the other of the ν derivatives of this class are linear functions, so that they vanish when these $(n-r)$ vanish, and let U be expressed as a function of $(d_1, d_2 \ldots d_{n-r}; x_1, x_2 \ldots x_r)$. Then we may write

$$U = \phi_{m,0} + \phi_{m-1,1} + \phi_{m-2,2} + \ldots + \phi_{1,m-1} + \phi_{0,m},$$

where in general $\phi_{m-\epsilon,\epsilon}$ denotes a function homogeneous and of $m-\epsilon$ dimensions in respect to $d_1, d_2 \ldots d_{n-r}$, and homogeneous and of ϵ dimensions in respect to $x_1, x_2 \ldots x_r$. Now the linear derivatives of U all vanish when $d_1 = 0$, $d_2 = 0 \ldots d_{n-r} = 0$ for all values of $x_1, x_2 \ldots x_r$. Hence $U = 0$ on the same supposition, and hence $\phi_{0,m}$ is similarly zero. Also the first derivatives of U, *quâ* $d_1, d_2 \ldots d_{n-r}$, must vanish on the same supposition. Hence $\phi_{1,m-1}$ is identically zero; and so by taking the 2nd, 3rd ... up to the $(m-1)$th or linear derivatives of U in respect to $d_1, d_2 \ldots d_{n-r}$, we find successively $\phi_{2,m-2}, \phi_{3,m-3} \ldots \phi_{m-1,1}$ each identically zero, and consequently

$$U = \phi_{m,0} = \phi(d_1, d_2 \ldots d_{n-r}),$$

as was to be proved. To express the fact of the ν derivatives being linear functions of $(n-r)$ of them, form a rectangular matrix with the coefficients of the ν linear derivatives. This matrix will be n terms in breadth and ν terms in depth. Let $r=1$: it is a direct consequence of the rule which has been established, that every full determinant consisting of a square n terms by n terms that can be formed out of this rectangular matrix must be zero: again, let $r=2$; all the first minors, that is to say, all the determinants composed of squares $(n-1)$ terms by $(n-1)$ terms, must be zero, and so in general a loss of r orders will require that the $(r-1)$th minors shall all vanish; if $r=n$, the $(n-1)$th minors, that is the simple terms of the matrix which are all coefficients of U, must vanish, or in other words, when the function is of zero order all the coefficients vanish (an obvious truism).

Thus, then, we see that the true rule for the loss of one order in a polynomial of any degree is precisely the same as the well-known rule for the loss of one order in a quadratic function; the speciality in the latter case consisting merely in the fact that ν being equal to n, the rectangular matrix becomes a square, and there is only one full determinant. Moreover, for any other value of r the above rule coincides with that given by me some time back in the *Philosophical Magazine* for the case of quadratic functions.

Professor Hesse's rule for finding conditions applicable to the loss of one order is, as I have already stated, a consequence of the more simple scheme of conditions above given. It consists in forming the determinant

$$\begin{vmatrix} \dfrac{d^2U}{dx_1^2}, & \dfrac{d^2U}{dx_1dx_2} \cdots & \dfrac{d^2U}{dx_1dx_n} \\ \dfrac{d^2U}{dx_2dx_1}, & \dfrac{d^2U}{dx_2dx_2} \cdots & \dfrac{d^2U}{dx_2dx_n} \\ \cdots & \cdots & \cdots \\ \dfrac{d^2U}{dx_ndx_1}, & \dfrac{d^2U}{dx_ndx_2} \cdots & \dfrac{d^2U}{dx_ndx_n} \end{vmatrix},$$

and equating the coefficients of this determinant fully developed separately to zero*. The attachment of the Professor to this particular form of covariant (I use the language of the calculus of forms) is readily intelligible, seeing the admirable application which he has made of it to the canonization of the cubic function of three variables, but it is really foreign to the nature of the present question; the coefficients of this covariant may easily be shown to be merely the full determinants of the $n \times \nu$ rectangular matrix above described, or linear functions of these said determinants with numerical coefficients. Hence the ground of its applicability.

Returning to the rule of the matrix, if we suppose the number of variables to be two, and call the coefficients of U

$$a_0,\ na_1,\ \tfrac{1}{2}n(n-1)a_2 \ldots a_n,$$

our rectangle becomes

$$\begin{vmatrix} a_0, & a_1 \\ a_1, & a_2 \\ a_2, & a_3 \\ \cdots & \cdots \\ \cdots & \cdots \\ a_{n-1}, & a_n \end{vmatrix}$$

* A form capable of being so derived I have elsewhere termed (in compliment to M. Hesse) the Hessian of the function to which it appertains. This is the trivial name which is much needed on account of the frequent occurrence of the form, and has been adopted by Mr Salmon

and the conditions become

$$a_0a_2 - a_1^2 = 0, \qquad a_0 a_3 - a_1 a_2 = 0,$$
$$a_1 a_3 - a_2^2 = 0, \qquad \dots\dots\dots\dots$$
$$\dots\dots\dots\dots \qquad \dots\dots\dots\dots$$
$$a_{n-2}a_n - a^2_{n-1} = 0, \quad a_{n-3}a_n - a_{n-2}a_{n-1} = 0, \text{ \&c.},$$

all of which equations are obviously true (when the function loses an order, that is to say, becomes a perfect power) and are satisfied (special cases excepted) when any $(n-1)$ independent equations out of the entire number obtain; so that the number of conditions implied in the property to be represented is in exact conformity with the number of independent equations derived from the matrix, that is equations which, when satisfied, will in general cause all the rest to be satisfied. This conformity manifests itself also in the case of a quadratic function of n variables. But except in these two limiting (and, in an occult sense, reciprocal*) cases of a function of two variables of the nth degree, or of the degree 2 and n variables, this conformity in measure as the degree or number of variables rises, although it must substantially continue to exist, becomes, and in an accelerated degree, less and less apparent.

Thus, take the simple case of a cubic function of three variables, and let us confine ourselves to the consideration of the conditions which must be satisfied when it loses a single order. Let U be written out at length,

$$ax^3 + by^3 + cz^3 + 3hyz^2 + 3izx^2 + 3jxy^2 + 3h'y^2z + 3i'z^2x + 3j'x^2y + 6mxyz.$$

in his admirable treatise on the higher plane curves. In systematic nomenclature it would be termed the discriminant of the quadratic emanant, or more briefly, the quadremanative discriminant. I have discovered quite recently that the long sought for symmetrical, and by far the most easy practical process for discovering the number of the real roots of an equation, is contained in, and may be deduced immediately from, a certain transformation of its Hessian!

* There are frequent cases occurring in the calculus of forms of interchange between the degree of a function and the number of variables which it contains. Thus, to select a striking example (although one where the interchange is not exact), the theory of the real and imaginary roots or factors of a homogeneous function of two variables and of the nth degree may be shown to be immediately dependent upon the determination of the specific nature of a concomitant homogeneous function of the 2nd degree and of $(n-1)$ variables. For instance, if any ordinary algebraical equation of the 5th degree be given, a homogeneous quadratic function of four variables may be constructed, representing, consequently, a surface of the 2nd degree [the coefficients of which (as indeed is true whatever be the degree of the equation) will be quadratic functions of the coefficients of the given equation]; and such that, according as the surface so represented belongs to the class of (1), impossible surfaces; (2), the ellipsoid or hyperboloid of two sheets; (3), the hyperboloid of one sheet; the given equation will have 5, 3, or only 1 real root! Moreover, an equality between two of the roots of the equation will be denoted by the loss of one order in the associated quadratic function; and so many orders altogether will be lost as there are independent equalities existing between the roots. An entirely new light is thus thrown on M. Sturm's theorem; and the number of real and imaginary roots in an equation is for the first time made to depend upon the signs of functions symmetrically constructed in respect to the two ends of the equation, which has long been felt as a desideratum.

The matrix formed out of the coefficients of the linear derivatives becomes

$$\begin{vmatrix} a, & j', & i \\ j, & b, & h' \\ i', & h, & c \\ m, & h', & h \\ i, & m, & i' \\ j', & j, & m \end{vmatrix}.$$

Now by the homaloidal law, if the terms in this rectangle were all unlike, the number of full determinants (3 terms by 3 terms) whose evanescence (except for special values) determines the evanescence of all the rest, should be $(6-3+1)(3-3+1)$, that is 4; but in the actual case, since the evanescence of all the full determinants is a necessary consequence of the function becoming a cubic function of two orders (that is, breaking up into the product of three linear functions of x, y, z), and as this decomposability, as is well known, implies only the existence of three affirmative conditions, the four full determinants

$$\begin{vmatrix} a, & j', & i \\ j, & b, & h' \\ i', & h, & c \end{vmatrix} \quad \begin{vmatrix} a, & j', & i \\ j, & b, & h' \\ m, & h', & h \end{vmatrix} \quad \begin{vmatrix} a, & j', & i \\ j, & b, & h' \\ i, & m, & i' \end{vmatrix} \quad \begin{vmatrix} a, & j', & i \\ j, & b, & h' \\ j', & j, & m \end{vmatrix} *$$

* That is to say, a syzygetic relation must connect these four determinants. I may as well here repeat, that when the vanishing of a set of i rational integral functions necessarily, and without cases of exception, implies the vanishing of another rational integral function, then this function is termed a syzygetic function of the others; and some power of it must be expressible under the form of a sum of i binary products of rational integral functions, one factor of each of which products must be one of the i given functions. When the vanishing of all but one of a set of functions in general necessarily implies the vanishing of that one, but subject to cases of exception for specific values of the variables, then it can only be affirmed that the functions of the set are in syzygy; that is to say, that the sum of the products of each of them respectively by some rational integral function will be zero: the equation expressing this relation is termed a syzygetic equation.

Thus, if we take the three full determinants that can be formed out of the matrix

$$\begin{matrix} a, & \alpha, \\ b, & \beta, \\ c, & \gamma, \end{matrix}$$

that is $$a\beta - b\alpha, \quad b\gamma - c\beta, \quad c\alpha - a\gamma,$$

these are in syzygy, for we can form the equation

$$c(a\beta - b\alpha) + a(b\gamma - c\beta) + b(c\alpha - a\gamma) = 0.$$

This, however, is not the only equation of the kind that can be formed, for

$$\gamma(a\beta - b\alpha) + \alpha(b\gamma - c\beta) + \beta(c\alpha - a\gamma) = 0$$

is also identically true. We see in this case that the evanescence of any two of the three functions which in the general case would be entirely independent, in this case cease to be so; and the vanishing of three of them must draw along with it by necessary implication (except for special values) the evanescence of the 4th, for thus only can the necessary conformity between the number of affirmative conditions and the number of unimplicated equations come to take effect. The clear and direct putting in evidence of this peculiar species of implication demands and deserves to be minutely considered; and as it must in part borrow its explanation from the very little yet known of syzygetic relations, so it must also throw new light on that great and important, but as yet unformed and scarcely more than nascent theory.

In conclusion, it is apparent from the demonstration above given, that when U, a function of n variables, becomes expressible as a function of $(n-r)$ orders, these orders may be taken respectively any independent linear functions of the linear derivatives of U, which remark completes the theory of functions subject to the loss of one or more orders. It is obvious (and I am indebted to my esteemed friend Mr Cayley for the remark), that the conditions furnished as above by the $(m-1)$th, that is linear derivatives, are identical with and may be more elegantly replaced by those involved in the assertion of the existence of linear relations between the 1st or $(m-1)$th degreed derivatives, and we have then this very simple rule; *if ϕ, a function of $x_1, x_2 \ldots x_n$, is expressible as a function of $n-r$ linear functions of $x_1, x_2 \ldots x_n$, it is necessary and sufficient that r independent linear relations shall exist between*

$$\frac{d\phi}{dx_1}, \frac{d\phi}{dx_2} \cdots \frac{d\phi}{dx_n}.$$

$a\beta - b\alpha$; $b\gamma - c\beta$; $c\alpha - a\gamma$ will in general imply the third, subject, however, to special cases of exception. Thus, if the 1st and 2nd vanish, the 3rd must vanish unless b and β *both* vanish; if the 2nd and 3rd vanish, the 1st must vanish unless c and γ both vanish; if the 3rd and 1st vanish, the second will vanish unless a and α both vanish. It will thus be seen that a peculiar species of *astricted* syzygy obtains between the three proposed functions, which enables us to affirm that *in general*, and except under *extra special* conditions, all three must vanish simultaneously. If two out of the three vanish, and the 3rd does not vanish, it is not merely (as might at the first blush of the theory of syzygy be conjectured) because some one other function vanishes in its place, but necessarily because a *plurality* of entirely independent functions (two simple letters as it happens here) each separately vanish. Thus we see how all but one of a set of functions $\chi_1, \chi_2 \cdots \chi_n$ may *in general*, and yet *not universally*, necessarily vanish when all the rest vanish: to say that *one* syzygetic equation such as

$$\chi_1\chi_1' + \chi_2\chi_2' + \cdots + \chi_n\chi_n' = 0$$

obtains, is not enough to explain the circumstances of the case; the fact is, that several distinct systems of values of $\chi_1', \chi_2' \ldots \chi_n'$ will be found capable of satisfying the equation, so that each of the functions $\chi_1, \chi_2 \cdots \chi_n$ will have a *system* of syzygetic factors attached to it, and these unrelated, in the wide sense that, if we take χ_n', χ_n'', any two of the syzygetic factors attached to χ_n, they will *not be in syzygy* with $\chi_1, \chi_2 \cdots \chi_{n-1}$; so that when these $(n-1)$ functions vanish, the vanishing of χ_n' and χ_n'' represents two distinct and completely independent conditions. Thus, in fine, the mutual implication of functions will in general denote the possibility of forming a *series* of syzygetic equations between them,—a remark, this, of no minor importance.

This rule itself also, it is evident, is capable of an independent and immediate demonstration by means of integrating the partial differential equation or equations by which it admits of being expressed. The above theory may readily be extended to functions of several systems of variables. Thus, for instance, the determinant

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

vanishing will be indicative of the function

$$\left\{\begin{array}{l} a\ xu + b\ xv + c\ xw \\ + a' yu + b' yv + c' yw \\ + a'' zu + b'' zv + c'' zw \end{array}\right\},$$

being linearly equivalent to a function of the form

$$\left\{\begin{array}{l} Ax'u' + Bx'v' \\ + Cy'u' + Dy'v' \end{array}\right\},$$

that is losing an order in respect of each of the two systems x, y, z; u, v, w; and so in general.