59.

ON MR CAYLEY'S IMPROMPTU DEMONSTRATION OF THE RULE FOR DETERMINING AT SIGHT THE DEGREE OF ANY SYMMETRICAL FUNCTION OF THE ROOTS OF AN EQUATION EXPRESSED IN TERMS OF THE COEFFICIENTS.

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FOR a considerable time past, among the few cultivators of the higher algebra, a proposition relative to the theory of the symmetrical functions of the roots of an equation has been in private circulation, which, to say nothing of the important applications of which it has been found susceptible to the calculus of forms, merits (by reason of its extreme simplicity), although, strange to say, it has, I believe, not yet obtained, a place in elementary treatises on algebra. The proposition alluded to I have reason to think first came to be observed in connexion with my well-known formulæ for Sturm's auxiliary functions in terms of the roots given in this *Magazine*. The theorem is briefly as follows. If a, b, c, &c. be the roots of an equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \&c. = 0,$$

any symmetric function such as $\Sigma a^\alpha b^\beta c^\gamma \ldots$, where α, β, $\gamma \ldots$ are positive integers arranged according to the order of their magnitudes in a descending (or, to speak more strictly, non-ascending) order, when expressed as a function of the coefficients, will be made up of terms of the form $p_1^{\theta_1} p_2^{\theta_2} p_3^{\theta_3} \ldots p_k^{\theta_k}$, such that $\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_k$ will be equal to α for some terms, but will for no term exceed α; α being, as above described, that one of the indices α, β, $\gamma \ldots$ which is not less than any of the others.

I had prepared, and indeed despatched, a somewhat elaborate proof of this theorem for the *Cambridge and Dublin Mathematical Journal*; but on proceeding to explain my method to Mr Cayley, elicited from that sagacious analyst the following excellent impromptu, which I think too valuable to be lost; and as it is now a twelvemonth or two since our conversation on the subject took place, and the author has not cared to put it on record, I feel myself under an obligation so to do, the more so as it entirely supersedes the comparatively inelegant demonstration of my own which I had previously intended to publish.

The method rests essentially on the following well-known theorem given by Euler relative to the partition of numbers; to wit, that the number of ways of breaking up a number n into parts is the same, whether we impose the condition that the number of parts in any partitionment shall not exceed m, or that the magnitude of any one of the parts shall not exceed m. Of this rule more hereafter—for the present to its application to the matter in hand.

Since $a, b, c \dots$ are the roots of $x^n + p_1 x^{n-1} + \dots$, we have

$$p_1 = a + b + c + \dots$$

$$p_2 = ab + ac + bc + \dots$$

$$p_3 = abc + abd + acd + \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Let $\alpha + \beta + \gamma + \dots = n$, none of the quantities $\alpha, \beta, \gamma \dots$ being greater than m, but $\alpha, \beta, \gamma \dots$ being otherwise arbitrary and capable of becoming equal to any extent *inter se*. Also let $\lambda + \mu + \nu + \dots = n$, the number of quantities λ, μ, ν, &c. being never greater than m, but the quantities themselves being otherwise arbitrary, and being capable of becoming equal to any extent *inter se*. By Euler's rule the number of systems $\alpha, \beta, \gamma \dots$ is the same as of the systems $\lambda, \mu, \nu \dots$, say P for each. For any system $\lambda, \mu, \nu \dots$, we shall have $p_\lambda p_\mu p_\nu \dots$, by virtue of the equations above written, expressible as the sum of terms of the form $\Sigma a^\alpha b^\beta c^\gamma \dots$; it may easily be made ostensible, that *all* the combinations of $\alpha, \beta, \gamma \dots$ subject to the above prescribed conditions must come into evidence by giving $\lambda, \mu, \nu \dots$ all the variations of which they admit; but this is also immediately obvious indirectly from the consideration, that were it otherwise, linear relations would subsist between the different values of $p_\lambda p_\mu p_\nu \dots$, which is obviously absurd. Hence, then, we shall be able to express the P quantities of the form $p_\lambda p_\mu \dots$ by means of linear functions of the P quantities $\Sigma a^\alpha b^\beta c^\gamma \dots$; and conversely, by solving the linear equations thus arising, the P quantities $\Sigma a^\alpha b^\beta c^\gamma \dots$ may be expressed in terms of the quantities $p_\lambda p_\mu \dots$; consequently $\Sigma a^m b^\beta c^\gamma \dots$, where m is greater or not less than any of the quantities $\beta, \gamma \dots$, will be expressible by means of combinations $p_\lambda p_\mu \dots$, where the number of coefficients $p_\lambda p_\mu \dots$ (any number of which may become identical) is for some of the combinations as great as, but for none of the combinations greater than m, as was to be proved. It will of course be seen that, for the purposes of the demonstration above given, it would have been sufficient to have been able to assume that the number of partitions, when the greatest part is not allowed to exceed m, is not *greater* than the number of partitions when the number of parts in any one partitionment does not exceed m. The equality of these two numbers would then evince itself in the course of the demonstration as a consequence of this assumption.

A word now as to Euler's beautiful law upon which the above demonstration is based.

A corollary from it, obtained by subtracting the equation which it gives when the limiting number is taken $(m-1)$ from the equation which it gives when the limiting number is m, will be the following proposition. The number of modes of partitioning n into m parts is equal to the number of modes of partitioning n into parts, one of which is always m, and the others m or less than m. This proposition was mentioned to me by Mr N. M. Ferrers*, whose demonstration of it (probably not different from that of Euler's for the other proposition, of which it may be viewed as a corollary) is so simple and instructive, that I am sure every logician will be delighted to meet with it here or elsewhere. It affords a most admirable example of that rather uncommon kind of reasoning whereby two abstract integers are proved to be equal indirectly, by showing that neither can be greater than the other.

If there be a group of A's and a group of B's, and every A can be shown to produce a B, and every B can be shown to produce an A, no matter whether the A producing a B is the same as, or different from, the A produced by that B, it is obvious that the number of A's cannot exceed that of the B's, nor of the B's that of the A's, and the two numbers will *therefore* be equal.

Take any such grouping as 3, 3, 2, 1, say A. This may be written as

$$\begin{array}{lll} 1, & 1, & 1 \\ 1, & 1, & 1 \\ 1, & 1, & \\ 1, & & \end{array}$$

and by reading off the columns as lines, may be transformed into the group

$$\begin{array}{llll} 1, & 1, & 1, & 1 \\ 1, & 1, & 1 & \\ 1, & 1 & & \end{array}$$

that is 4, 3, 2, say B.

* I learn from Mr Ferrers that this theorem was brought under his cognizance through a Cambridge examination paper set by Mr Adams of Neptune notability.

In A the number of parts is 4. In B the greatest part is 4; the others might be (although they happen not in this particular instance to be) 4, but cannot be greater than 4. And so every A in which the number of parts is 4 will give rise to a B in which 4 is one of the parts, and every other part is 4 or less, and evidently (although, as above remarked, this is immaterial to the demonstration) every such B gives reciprocally the same A from which it is itself derived; hence the number of A's and B's is equal. This is the theorem which, for the sake of distinction, I have called the Corollary to Euler's. Euler's own is proved by the same diagram; for if we define A as a grouping where the number of parts *does not exceed* 4, we get a definition of B as a grouping where the greatest part does not exceed 4, and so in general. We see that this theorem may be varied also by affirming that the number of ways in which n may be broken up, so that there shall never be less than m parts, is the same as the number of ways in which it may be broken up into parts, the greatest of which in any one way is not less than m. So, again, a similar diagram makes it apparent, that if we break up each of i numbers into parts so that the sum of the greatest parts shall not exceed (or be less than) m, the number of ways in which this can be done will be the same as the number of ways in which these i numbers can be simultaneously partitioned so that the total number of parts in any simultaneous partitionment shall never exceed (or never be less than) m; and doubtless an extensive range of analogous general theorems relative to the partitioning of numbers may be struck out by aid of the same diagram, by no means easily demonstrable unless this simple mode of conversion happen to be thought of, but in that event becoming intuitively apparent. This mode of conversion is precisely that (only applied to a more general state of things) whereby, in elementary arithmetic, it is established that m times n is the same as n times m. A consideration of the process by which the mind satisfies itself of the universality of this law, has been always sufficient to convince me of the absurdity of ascribing to an inductive process the capacity of the human mind for forming general ideas concerning necessary relations.