61.

ON A REMARKABLE MODIFICATION OF STURM'S THEOREM.

[*Philosophical Magazine,* V. (1853), pp. 446—456.]

LET me be allowed to use the term *improper* continued fraction to denote a fraction differing from an ordinary continued fraction, in the sole circumstance of the numerators being all negative units instead of positive units, as thus:

$$\cfrac{1}{q_1 - \cfrac{1}{q_2 - \cfrac{1}{q_3 - \text{\&c.}}}}$$

The successive convergents of such a fraction as that written above will be

$$\frac{1}{q_1}, \quad \frac{q_2}{q_2q_1 - 1}, \quad \frac{q_3q_2 - 1}{q_3q_2q_1 - q_3 - q_1}, \text{ \&c.}$$

If we call these respectively

$$\frac{N_1}{D_1}, \quad \frac{N_2}{D_2}, \quad \frac{N_3}{D_3}, \text{ \&c.}$$

we have the general scale of formation

$$N_\iota = q_\iota N_{\iota-1} - N_{\iota-2},$$
$$D_\iota = q_\iota D_{\iota-1} - D_{\iota-2}.$$

Moreover, we shall have universally

$$N_\iota D_{\iota-1} - N_{\iota-1} D_\iota \text{ equal to } +1,$$

instead of alternating between $+1$ and -1, as is the case in continued fractions of the ordinary kind.

Again, let me be allowed to use the term *signaletic* series to denote a series of disconnected terms, designed to exhibit a certain succession of algebraical signs + and −, and to speak of two series being *signaletically* equivalent when the *number* of continuations of signs and of variations of signs between the several terms and those that are immediately contiguous to them is the same for the two series; a condition which evidently may be satisfied without the *order* of such changes and continuations being identical. I am now able to enunciate the following remarkable *theorem of signaletic equivalence* between two distinct series of terms, each generated from the same improper continued fraction. But first I must beg to introduce yet another new term in addition to those already employed, namely *reverse convergents*, to denote the convergents generated from a given continued fraction by reading the quotients in a reverse order, or if we like so to say, the convergents corresponding to the given continued fraction reversed.

The two forms

$$\frac{1}{q_1 - \frac{1}{q_2 - \frac{1}{q_3 \atop \vdots - \frac{1}{q_n}}}} \quad \text{and} \quad \frac{1}{q_n - \frac{1}{q_{n-1} - \frac{1}{q_{n-2} \atop \vdots - \frac{1}{q_1}}}},$$

are obviously reciprocal; and if the two last convergents of either one of them be respectively

$$\frac{N_{n-1}}{D_{n-1}}, \ \frac{N_n}{D_n},$$

$\frac{D_{n-1}}{D_n}$ will serve to generate the other. For the clearer and more simple enunciation of the theorem about to be given, it will be better to take as our first convergent $\frac{0}{1}$, so that 1 will be treated as the denominator of the first convergent in every case; and calling D_0 such denominator, we shall always understand that $D_0 = 1$. Let now $D_0, D_1, D_2 \ldots D_n$ be the $(n+1)$ denominators of any improper continued fraction of n quotients, and $\mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2 \ldots \mathfrak{D}_n$ the corresponding denominator series for the same fraction reversed; then, I say, that these *two series* are *signaletically equivalent.*

I do not here propose to demonstrate this proposition, to which I was led unconsciously by researches connected with the theory of elimination, which afford a complete and general but somewhat indirect and circuitous proof. Doubtless some simple and direct proof cannot fail ere long to be discovered*. For the present I shall content myself with showing *à posteriori* the truth of the theorem for a particular case. Let $n = 3$. The two series which are to be proved to be signaletically equivalent may be written

$$1, \ A, \ BA - 1, \ CBA - C - A,$$

$$1, \ C, \ BC - 1, \ ABC - A - C.$$

* See Postscript [p. 616 below].

Call these respectively S and (S). In S we may substitute in the third term, in place of $BA-1$, CA without affecting the signaletic value of the series; for if the second and fourth terms have different signs, the third term may be taken anything whatever, since the sequence of the second, third, and fourth terms will give one continuation and one change, whatever the middle one may be. Suppose, then, that the second and fourth terms have the same sign, and let

$$CBA - C - A = m^2A,$$

therefore $$C(BA-1) = (m^2+1)A,$$

therefore $$(BA-1)AC = (m^2+1)A^2.$$

Hence $BA-1$ and AC will have the same sign; hence S is signaletically equivalent to S', where S' denotes the series

$$1,\quad A,\quad CA,\quad CBA - C - A.$$

Now, again, if CA is negative, we may put instead of A anything whatever, and therefore, if we please, C, without affecting signaletically the value of S'. But if CA is positive, A and C will have the same sign, and therefore on this supposition also C may be substituted for A. Hence always S' is signaletically equivalent to S'', where S'' denotes

$$1,\quad C,\quad CA,\quad CBA - C - A.$$

Again, if C and $CBA - C - A$ have different signs, the value of the intermediate term is immaterial; but if C and $CBA - C - A$ have the same sign, let

$$CBA - C - A = m^2C;$$

then $$A(CB-1) = (1+m^2)C,$$

and $$A^2(CB-1) = (1+m^2)AC;$$

and consequently $CB-1$ and AC have the same sign. In every case, therefore, S'' is signaletically equivalent to

$$1,\quad C,\quad CB-1,\quad ACB - A - C;$$

that is S is signaletically equivalent to S', and therefore to S'', and therefore to (S), as was to be proved.

The application of the foregoing theory to Sturm's process for finding the number of real roots of an equation is apparent; for a very little consideration will serve to show, that if we expand $\frac{f'x}{fx}$, fx being of the nth degree in x, algebraically under the form of a continued fraction

$$\cfrac{1}{Q_1 - \cfrac{1}{Q_2 - \cfrac{1}{Q_3 \; \vdots - \cfrac{1}{Q_n}}}},$$

where Q_1, Q_2, $Q_3 \ldots Q_n$ may be supposed linear functions of x (although, in fact, this restriction, as will be hereafter noticed, is unnecessary), the denominators of the reverse convergents

$$\frac{0}{1},\ \frac{1}{Q_n},\ \frac{Q_{n-1}}{Q_n Q_{n-1} - 1} \cdots \frac{Q_{n-1} Q_{n-2} \ldots Q_1 - \&c.}{Q_n Q_{n-1} \ldots Q_1 - \&c.},$$

will be signaletically equivalent with the Sturmian series of functions for determining the number of real roots of fx within given limits; in fact,

$$1,\quad Q_n,\quad Q_n Q_{n-1} - 1, \ldots , Q_n Q_{n-1} \ldots Q_1 - \&c.$$

will be the Sturmian functions themselves, divided out by the negative of the last or constant residue which arises in the application of the process of continued division, according to Sturm's rule; and as we have shown that the series of the denominators to the convergents of any continued fraction, and the series of the denominators to the convergents of the same fraction reversed, are signaletically equivalent, we have this surprisingly new, interesting, and suggestive mode of stating Sturm's theorem, namely, the denominators to the convergents of the continued fraction which represents $\frac{f'x}{fx}$ constitute a Rhizoristic series for fx, that is a signaletic series which serves to determine the number of roots of fx comprised within any prescribed limits. Moreover, in applying this theorem it is by no means necessary that, in the continued fraction which represents $\frac{f'x}{fx}$, all or any of the quotients should be taken linear functions of x. A very little consideration of the principles upon which the demonstration of Sturm's theorem is founded will serve to show that the convergent denominators to any continued fraction whatever which represents $\frac{f'x}{fx}$, whether the quotients be linear or non-linear, integral or fractional, or mixed functions of x, and whatever the number of quotients, which, it may be observed, cannot be less than, but may be made to any extent greater than the exponent of the degree of fx, will equally well furnish a Rhizoristic series for fixing the position of the roots, provided only that the last *divisor* in the process of expanding $\frac{f'x}{fx}$ under the form of an improper continued fraction be a constant quantity or any function of x incapable of changing its sign.

Let us, however, for the present confine our attention to the ordinary Sturmian form, where all the quotients are linear functions of x. Let these quotients be respectively

$$a_1 x + b_1,\quad a_2 x + b_2,\quad a_3 x + b_3 \ldots a_n x + b_n.$$

In order to determine the total number of real and imaginary roots of fx, we must count the loss of continuations of sign in the Rhizoristic series in passing from $x = +\infty$ to $x = -\infty$. When x is infinitely great, it is clear that, whether positive or negative, the parts $b_1, b_2 \dots b_n$ may be neglected, and only the highest powers of x need be attended to in writing down the signaletic series corresponding to these two values of x. Accordingly for $x = \pm\infty$ the signaletic series becomes

$$1,\ a_1x,\ a_1a_2x^2,\ \dots,\ a_1a_2\dots a_nx^n,$$

and consequently the number of pairs of imaginary roots of fx is the number of changes of sign in the series

$$1,\ a_1,\ a_1a_2,\ \dots,\ a_1a_2\dots a_n,$$

that is, is the number of negative quantities in the series

$$a_1,\ a_2,\ a_3,\ \dots,\ a_n.$$

Hence we have the curious and hitherto strangely overlooked theorem, that in applying Sturm's process of successive division to fx and $f'x$, the number of negative coefficients of x in the successive quotients gives the number of pairs of imaginary roots of fx; as a corollary, we learn the somewhat curious fact that never more than half of these coefficients can be negative; and in general it would appear that the better practical method of applying Sturm's theorem would be not to deal with the Residues, which have hitherto been the sole things considered, but rather with the linear quotients which have been treated as merely incidental to the formation of the Residues.

To find the value of the Rhizoristic series corresponding to a given value of x, the better method would accordingly seem to be to commence with finding the arithmetical values of the n quotients

$$a_1x + b_1,\ a_2x + b_2 \dots a_nx + b_n.$$

We thus obtain n numbers $\mu_1, \mu_2 \dots \mu_n$, and have only to form a progression according to the well-known law

$$1,\ N_1,\ N_2 \dots N_n,$$

where $N_1 = \mu$, and in general $N_\iota = \mu_\iota N_{\iota-1} - N_{\iota-2}$.

The number of arithmetical operations required by this method (after the division part of the process which is common to the two methods has been performed) will be* $2n$ multiplications and $2n$ additions or subtractions; whereas if we deal with the residues directly, the number of multiplications will be

$$n + (n-1) + \dots + 1,$$

that is $$\frac{n(n+1)}{2},$$

(besides having to raise x to the nth power), and the same number of additions. The practical advantage, however, of this method over the old

[* *footnote*, p. 622 below.]

method is not quite so great as it may at first sight appear, in consequence of the quantities operated with on applying it being larger numbers than those which have to be used in the old method.

If we were to employ, instead of the direct series,

$$1,\ N_1,\ N_2N_1-1,\ \&c.,$$

the signaletically equivalent reverse series

$$1,\ N_n,\ N_{n-1}N_n-1,\ \&c.,$$

the arithmetical difficulty would be much increased in consequence of the quotients becoming rapidly more complex as the division proceeds. It were much to be desired that some person practically conversant with the application of Sturm's method, such as that excellent and experienced mathematician, my esteemed friend Professor J. R. Young, would perpend and give his opinion upon the relative practical advantages of the two methods of substitution; the one that where the residues are employed, the other that where the quotients.

I am bound to state, that but for a valuable hint furnished to me by my friend, that most profound mathematician, M. Hermite, who discovered a theorem virtually involving the transformation of Sturm's theorem here presented, but founded upon entirely different and less general considerations, and in the origin of which hint, as arising out of my own previous speculations upon which I was in correspondence with M. Hermite, I may perhaps myself claim a share, this theory would probably not have come to light. It is of course not confined to Sturm's theorem, which deals only with the special case of two functions, whereof one is the first derivative of the other.

There is a larger theory, to which M. Sturm's is a corollary, which contemplates the relations of the roots of any two functions whatever. This is what I term the *theory of interpositions*, upon which I do not propose here to enter, but which will be fully developed in a memoir nearly completed, and which I shortly propose to present* to the Royal Society, wherein will be found combined and flowing into one current various streams of thought bearing upon this subject which had previously existed disunited, and appearing to follow each a separate course.

Remark.

I am not aware that anyone has observed what the effect would be of omitting to change the signs of the successive residues in the application of Sturm's method, that is, of employing a proper in lieu of an improper continued fraction to express $\frac{f'x}{fx}$.

[* pp. 429—586 above.]

Although easily made out, it is well worthy of being remarked. Suppose

$$\frac{\phi}{f} = \cfrac{1}{Q_1 - \cfrac{1}{Q_2 - \cfrac{1}{Q_3 \atop \vdots - \cfrac{1}{Q_n}}}},$$

and in general (P being any letter) use $\bar{P}$ to denote $-P$. Now we may write

$$\begin{aligned} f &= Q_1\phi - \rho_1, \\ \phi &= Q_2\rho_1 - \rho_2, \\ \rho_1 &= Q_3\rho_2 - \rho_3, \\ \rho_2 &= Q_4\rho_3 - \rho_4, \\ \rho_3 &= Q_5\rho_4 - \rho_5, \\ \rho_4 &= Q_6\rho_5 - \rho_6, \\ \&c. &= \&c. \end{aligned}$$

This gives

$$\begin{aligned} f &= Q_1\phi + \bar{\rho}_1, \\ \phi &= \bar{Q}_2\bar{\rho}_1 + \bar{\rho}_2, \\ \bar{\rho}_1 &= Q_3\bar{\rho}_2 + \rho_3, \\ \bar{\rho}_2 &= \bar{Q}_4\rho_3 + \rho_4, \\ \rho_3 &= Q_5\rho_4 + \bar{\rho}_5, \\ \&c. &= \&c. \end{aligned}$$

The law evidently being that the quotients change their sign alternately, that is in the 2nd, 4th, 6th, &c. places, and remain unaltered in the 1st, 3rd, 5th, &c. places; whereas the residues or excesses change their signs in the 1st and 2nd, 5th and 6th, 9th and 10th, &c., and remain unaltered in the 3rd and 4th, 7th and 8th, 11th and 12th, &c. places. The effect is, that if, in applying Sturm's method, we omit to change the signs of the remainders, and take as our signaletic series

$$fx,\ f'x,\ R_1,\ R_2,\ R_3 \ldots R_{n-1},$$

R_1, R_2, R_3, &c. being the successive unaltered residues, the signaletic index corresponding to any value of x instead of being the number of continuations in the above series, will become the number of continuations in going from a term in an odd place to a term in an even place *plus* the number of variations in going from a term in an odd place to a term in an even place.

If we adopt the quotient method, the rule will be simply to change the sign of the alternate quotients (beginning with the second) in forming the signaletic series.

As an artist delights in recalling the particular time and atmospheric effects under which he has composed a favourite sketch, so I hope to be excused putting upon record that it was in listening to one of the magnificent choruses in the 'Israel in Egypt' that, unsought and unsolicited, like a ray of light, silently stole into my mind the idea (simple, but previously unperceived) of the equivalence of the Sturmian residues to the denominator series formed by the reverse convergents. The idea was just what was wanting,—the key-note to the due and perfect evolution of the theory.

Postscript.

Immediately after leaving the foregoing matter in the hands of the printer, a most simple and complete proof has occurred to me of the theorem left undemonstrated in the text [p. 610].

Suppose that we have any series of terms $u_1, u_2, u_3 \ldots u_n$, where

$$u_1 = A_1,\quad u_2 = A_1A_2 - 1,\quad u_3 = A_1A_2A_3 - A_1 - A_3, \text{ \&c.}$$

and in general

$$u_\iota = A_\iota u_{\iota-1} - u_{\iota-2},$$

then $u_1, u_2, u_3 \ldots u_n$ will be the successive principal coaxal determinants of a symmetrical matrix. Thus suppose $n = 5$; if we write down the matrix

$$\begin{matrix} A_1, & 1, & 0, & 0, & 0, \\ 1, & A_2, & 1, & 0, & 0, \\ 0, & 1, & A_3, & 1, & 0, \\ 0, & 0, & 1, & A_4, & 1, \\ 0, & 0, & 0, & 1, & A_5, \end{matrix}$$

(the mode of formation of which is self-apparent), these successive coaxal determinants will be

$$1 \quad \begin{vmatrix} A_1 \end{vmatrix} \quad \begin{vmatrix} A_1, & 1 \\ 1, & A_2 \end{vmatrix} \quad \begin{vmatrix} A_1, & 1, & 0 \\ 1, & A_2, & 1 \\ 0, & 1, & A_3 \end{vmatrix} \quad \begin{vmatrix} A_1, & 1, & 0, & 0 \\ 1, & A_2, & 1, & 0 \\ 0, & 1, & A_3, & 1 \\ 0, & 0, & 1, & A_4 \end{vmatrix} \quad \begin{vmatrix} A_1, & 1, & 0, & 0, & 0 \\ 1, & A_2, & 1, & 0, & 0 \\ 0, & 1, & A_3, & 1, & 0 \\ 0, & 0, & 1, & A_4, & 1 \\ 0, & 0, & 0, & 1, & A_5 \end{vmatrix}$$

that is

$$1,\ A_1,\ A_1A_2 - 1,\ A_1A_2A_3 - A_1 - A_3,\ A_1A_2A_3A_4 - A_1A_2 - A_1A_4 - A_3A_4 + 1,$$
$$A_1A_2A_3A_4A_5 - A_1A_2A_5 - A_1A_4A_5 - A_3A_4A_5 - A_1A_2A_3 + A_5 + A_3 + A_1.$$

It is proper to introduce the *unit* because it is, in fact, the value of a determinant of zero places, as I have observed elsewhere. Now I have demonstrated directly in this very *Magazine* (August 1852)*, under cover of the *umbral* notation, that the *signaletic* value of a regularly ascending series of principal coaxal determinants formed from any symmetrical matrix is unaffected by any such transposition whatever of the lines and columns of the matrix as does not destroy the symmetry about the principal axis. Hence, then, beginning from the lower extremity of the axis A_5, and reading off the ascending series of coaxal minors from that point, we obtain the reverse series,

$$1,\ A_5,\ A_5A_4-1,\ A_5A_4A_3-A_5-A_3,\ A_5A_4A_3A_2-A_5A_4-A_5A_2-A_3A_2+1,$$
$$A_5A_4A_3A_2A_1-A_5A_4A_1-A_5A_2A_3-A_3A_2A_1-A_5A_4A_3+A_1+A_3+A_5.$$

Hence we see that the denominators to the convergents of

$$\cfrac{1}{A_1-\cfrac{1}{A_2-\cfrac{1}{A_3-\cfrac{1}{A_4-\cfrac{1}{A_5}}}}},$$

beginning with 1, form a series signaletically equivalent to that similarly formed from the fraction

$$\cfrac{1}{A_5-\cfrac{1}{A_4-\cfrac{1}{A_3-\cfrac{1}{A_2-\cfrac{1}{A_1}}}}};$$

and the reasoning is of course general, and establishes the theorem in question.

It seems only proper and natural that I should not leave unstated here the signaletic properties of the series of *numerators* to the convergents to $\dfrac{f'x}{fx}$ expanded under the form of a continued fraction.

Let the number of changes of sign in the denominator series for any given value a of x be called $D(a)$, and for the numerator series $N(a)$. Then $N(a)-N(b)$ may be equal to, or at most can only differ by a positive or negative unit from $D(a)-D(b)$. The relation between these differences depends on the nature of the interval between the greater of the two limits a and b, and the root of $f(x)$ next less than that limit, and of the interval between the less of the two limits a and b, and the root of fx next greater than such limit. If a root of $f'x$ is contained in each such interval,

$$N(a)-N(b)=D(a)-D(b)+1;$$

if a root of $f'x$ is contained within one interval, but no root within the other,

$$N(a)-N(b)=D(a)-D(b);$$

if no root of $f'x$ is contained within either interval,

$$N(a)-N(b)=D(a)-D(b)-1.$$

[* p. 380 above.]

I may conclude with noticing that the determinantive form of exhibiting the successive convergents to an improper continued fraction affords an instantaneous demonstration of the equation which connects any two consecutive such convergents as

$$\frac{N_{\iota-1}}{D_{\iota-1}} \text{ and } \frac{N_\iota}{D_\iota},$$

namely $$N_\iota D_{\iota-1} - N_{\iota-1} D_\iota = 1.$$

For if we construct the matrix, which for greater simplicity I limit to five lines and columns,

$$\begin{vmatrix} A, & 1, & 0, & 0, & 0 \\ 1, & B, & 1, & 0, & 0 \\ 0, & 1, & C, & 1, & 0 \\ 0, & 0, & 1, & D, & 1 \\ 0, & 0, & 0, & 1, & E \end{vmatrix}, \qquad \text{(M)}$$

and represent umbrally as

$$\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \end{pmatrix},$$

and if, by way of example, we take the fourth and fifth convergents, these will be in the umbral notation represented by

$$\frac{\begin{pmatrix} a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4 \end{pmatrix}}{\begin{pmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \end{pmatrix}} \text{ and } \frac{\begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix}}{\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \end{pmatrix}},$$

respectively. Hence

$$N_5D_4 - N_4D_5 = \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix} \times \begin{pmatrix} a_2, & a_3, & a_4, & a_1 \\ b_2, & b_3, & b_4, & b_1 \end{pmatrix}$$

$$- \begin{pmatrix} a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4 \end{pmatrix} \times \begin{pmatrix} a_2, & a_3, & a_4, & a_5, & a_1 \\ b_2, & b_3, & b_4, & b_5, & b_1 \end{pmatrix},$$

which [p. 252 above]

$$= \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix} \times \begin{pmatrix} a_2, & a_3, & a_4, & a_1 \\ b_2, & b_3, & b_4, & b_1 \end{pmatrix} - \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix} \begin{pmatrix} a_2, & a_3, & a_4, & a_1 \\ b_2, & b_3, & b_4, & b_1 \end{pmatrix}$$

$$+ \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_1 \end{pmatrix} \times \begin{pmatrix} a_2, & a_3, & a_4, & a_1 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix}$$

$$= \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4 \end{pmatrix} \times \begin{pmatrix} a_1, & a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix},$$

that is

$$\begin{vmatrix} 1, & B, & 1, & 0 \\ 0, & 1, & C, & 1 \\ 0, & 0, & 1, & D \\ 0, & 0, & 0, & 1 \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ B, & 1, & 0, & 0 \\ 1, & C, & 1, & 0 \\ 0, & 1, & D, & 1 \end{vmatrix} = 1 \times 1 = 1,$$

as was to be proved. And the demonstration is evidently general in its nature. We may treat a proper continued fraction in precisely the same manner, substituting throughout $\sqrt{(-1)}$ in place of 1 in the generating matrix, and we shall thus, by the same process as has been applied to improper continued fractions, obtain

$$N_{\iota+1}D_\iota - N_\iota D_{\iota+1} = \{\sqrt{(-1)}\}^\iota \times \{\sqrt{(-1)}\}^\iota$$
$$= (-1)^\iota.$$

I believe that the introduction of the method of determinants into the algorithm of continued fractions cannot fail to have an important bearing upon the future treatment and development of the theory of Numbers*.

* If in the above matrix (M) we write throughout $\sqrt{(-1)}$ in place of 1, we have a representation of the numerators and denominators of the convergents to a proper continued fraction, and such representation gives an immediate and visible proof of the simple and elegant rule (not stated in the ordinary treatises on the subject, nor so well known as it deserves to be) for forming any such numerators or denominators by means of the principal terms in each; the rule, I mean, according to which the ιth denominator may be formed from $q_1q_2q_3q_4 \dots q_\iota$ ($q_1, q_2 \dots q_\iota$ being the successive quotients), and the ιth numerator from $q_2q_3 \dots q_\iota$, by leaving out from the above products respectively any pair or any number of pairs of consecutive quotients as $q_\rho q_{\rho+1}$. For instance, from $q_1q_2q_3q_4q_5$, by leaving out q_1q_2, q_2q_3, q_3q_4 and q_4q_5, we obtain

$$q_3q_4q_5 + q_1q_4q_5 + q_1q_2q_5 + q_1q_2q_3;$$

and by leaving out $q_1q_2 \times q_3q_4$, $q_1q_2 \times q_4q_5$, $q_2q_3 \times q_4q_5$, we obtain $q_5 + q_3 + q_1$; so that the total denominator becomes

$$q_1q_2q_3q_4q_5 + q_3q_4q_5 + q_1q_4q_5 + q_1q_2q_5 + q_1q_2q_3 + q_1 + q_3 + q_5;$$

and in like manner the numerator of the same convergent is

$$q_2q_3q_4q_5 \left\{1 + \frac{1}{q_2q_3} + \frac{1}{q_3q_4} + \frac{1}{q_4q_5} + \frac{1}{q_2q_3q_4q_5}\right\},$$

that is
$$q_2q_3q_4q_5 + q_4q_5 + q_2q_5 + q_2q_3 + 1.$$

The most cursory inspection of the form of the generating matrix will show at once the reason of this rule. It may furthermore be observed, that every progression of terms constructed in conformity with the equation

$$u_n = a_nu_{n-1} - b_nu_{n-2} + c_nu_{n-3} \pm \text{\&c.},$$

may be represented as an ascending series of principal coaxal determinants to a common matrix. Thus if each term in such progression is to be made a linear function of the three preceding terms, it will be representable by means of the matrix

$$\begin{matrix} A, & B, & C'', & 0, & 0 \\ 1, & A', & B'', & C''', & 0 \\ 0, & 1, & A'', & B''', & C'''' \\ 0, & 0, & 1, & A''', & B'''' \\ 0, & 0, & 0, & 1, & A'''' \end{matrix}$$

indefinitely continued, which gives the terms

$$1,\ A,\ AA' - B,\ AA'A'' - BA'' - AB'' + C'',\ \text{\&c.}$$