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# NOTE ON A REMARKABLE MODIFICATION OF STURM'S THEOREM, AND ON A NEW RULE FOR FINDING SUPERIOR AND INFERIOR LIMITS TO THE ROOTS OF AN EQUATION.

[*Philosophical Magazine*, VI. (1853), pp. 14—20.]

In my paper [p. 609 above] on this subject in the preceding Number of the *Magazine*, I showed how by means of the quotients $a_1x+b_1,\ a_2x+b_2 \ldots a_nx+b_n$, obtained by throwing $\frac{f'x}{fx}$ under the form of a continued fraction, the process for finding the signaletic index for any given value of $x$ in the series for determining the number of real roots of $fx$ within given limits was reduced to performing two sets of $n$ multiplications and as many additions or subtractions. But by means of a very simple observation, I can now show that the second and more laborious set of multiplications may be dispensed with and replaced by the simple operation of finding reciprocals, which can be done by mere inspection by means of Barlow's or similar tables, which are familiar to all computers. If we call the quotients

$$a_1x + b_1,\ a_2x + b_2 \ldots a_nx + b_n,$$

we must, as explained in the preceding article, find the $n$ numerical values $\mu_1,\ \mu_2 \ldots \mu_n$ which these quotients assume for any assigned value of $x$. This being done, the signaletic index corresponding to such value of $x$, that is the number of continuations of sign in the signaletic series

$$1,\ \mu_1,\ \mu_1\mu_2 - 1,\ \mu_3\mu_2\mu_1 - \mu_3 - \mu_1,\ \&c.,$$

is evidently the number of positive terms in the series

$$\frac{1}{\mu_1},\ \frac{1}{\mu_2 - \cfrac{1}{\mu_1}},\ \frac{1}{\mu_3 - \cfrac{1}{\mu_2 - \cfrac{1}{\mu_1}}} \cdots \frac{1}{\mu_n - \cfrac{1}{\mu_{n-1} \, \vdots - \cfrac{1}{\mu_1}}}.$$

These terms may be found with the utmost facility in succession from one another; for if $M_\iota$ be one of them, the next will be $(\mu_{\iota+1} - M_\iota)^{-1}$. Thus, then, the necessity for the more operose set of multiplications is done away with, and the actual labour of computation reduced much more than 50 per cent. below that required by the method indicated in the preceding article on the subject. I need hardly add, that the old method of Sturm would admit of a similar abbreviation; but in using it we should be subjected to the great practical disadvantage of having to begin with the more heavy and complicated quotients $\mu_n$, $\mu_{n-1}$, &c. instead of $\mu_1$, $\mu_2$, &c., which would very greatly enhance the labour of computation. I will conclude by a remark of some interest under an algebraical point of view.

It has been stated that the denominators of the successive convergents to

$$\frac{1}{q_n} - \frac{1}{q_{n-1}} - \frac{1}{q_{n-2}} \text{ \&c.} \; \vdots \; - \frac{1}{q_1}$$

are equivalent (to a constant factor *près*) with the Sturmian functions, and the reader may be curious to know something of the nature of the signaletically equivalent series formed by the denominators of the convergents to the direct fraction

$$\frac{1}{q_1} - \frac{1}{q_2} - \frac{1}{q_3} \text{ \&c.} \; \vdots \; - \frac{1}{q_n}.$$

These denominators are (abstracting from a constant factor not affecting the signs) the Sturmian residues resulting from performing the process of common measure between $f'x$ and $f_1x$; $f_1x$ being related in a remarkable manner in point of form to $f'x$. Call the roots of $fx$ $a_1, a_2 \dots a_n$; we know that $f'x$ is

$$\Sigma \{(x - a_2)(x - a_3) \dots (x - a_n)\},$$

and I am able to state that $f_1x$ is (to a constant factor *près*) equal to

$$\Sigma [\zeta(a_2, a_3 \dots a_n) \{(x - a_2)(x - a_3) \dots (x - a_n)\}],$$

$\zeta(a_2, a_3 \dots a_n)$ denoting the product of the squares of the differences between the $(n-1)$ quantities $a_2, a_3 \dots a_n$. Accordingly it will be seen that whenever $x$ is indefinitely near, whether on the side of excess or defect, to a real root of $fx$, $f'x$ and $f_1x$ will have the same sign; which serves to show, upon an independent and specific algebraical ground, why the two series of residues corresponding to $\frac{f'x}{fx}$ and $\frac{f_1x}{fx}$ are (as by a deduction from a general principle they have been previously shown to be) *rhizoristically* equivalent.

*Observation.*

In comparing the relative merits of the old and new methods of substitution for the purposes of Sturm's theorem, the effect of the introduction of positive multipliers into the dividends in order to keep all the numerical quantities integral ought not to be disregarded. If we call the quotients corresponding to this modification of the dividends $Q_1$, $Q_2$, $Q_3$, $Q_4$, &c., and the factors thus introduced $m_1$, $m_2$, $m_3$, $m_4$, &c., the true quotients will be

$$\frac{Q_1}{m_1},\ \frac{m_1}{m_2}Q_2,\ \frac{m_2}{m_1 m_3}Q_3,\ \frac{m_1 m_3}{m_2 m_4}Q_4,\ \text{\&c.};$$

and it will be found that we may employ as our rhizoristic index either the number of continuations of sign in the series

$$1,\ Q_1,\ Q_2Q_1 - m_2,\ Q_3(Q_2Q_1 - m_2) - m_3Q_1,\ \text{\&c.}$$

the law of formation of the successive terms $u_0$, $u_1$, $u_2$, &c. being

$$u_{\iota+1} = Q_{\iota+1}u_\iota - m_{\iota+1}u_{\iota-1},$$

or the number of positive signs in the series

$$Q_1,\ Q_2 - \frac{m_2}{Q_1},\ Q_3 - \cfrac{m_3}{Q_2 - \cfrac{m_2}{Q_1}},\ \text{\&c.}$$

the law of formation of the successive terms $v_1$, $v_2$, $v_3$, &c. being

$$v_\iota = Q_\iota - \frac{m_\iota}{v_{\iota-1}}.$$

There may therefore, in fact, be in each case $(n-1)$ more multiplications than have been taken account of in the text above.

If integer numbers be used *throughout* (so that accordingly the $u$ series is that made use of), the total number of multiplications will in general be $n + 2(n-1)$* or $3n-2$; the old method, as previously stated, would require $\frac{1}{2}n(n+1)$ multiplications; for if we call any one of the Sturmian functions

$$A_0x^\iota + A_1x^{\iota-1} + A_2x^{\iota-2} + \ldots + A_\iota,$$

we shall, using the most abbreviated method of computation, have to calculate successively

$$xA_0 + A_1,\ x(xA_0 + A_1) + A_2,\ \text{\&c.},$$

* If all the extraneous factors are units, the number of multiplications (like that of the additions) would be $2n-1$, and not $2n$, as inadvertently stated in the preceding number of the *Magazine*.

giving rise to $\iota$ operations (but, it must be admitted, with the practical advantage of the use of a constant multiplier); and as $\iota$ may take all values from $n$ to 1, the total number of such operations will be $\frac{1}{2}n(n+1)$. When $n=4$,

$$\tfrac{1}{2}n(n+1)=3n-2.$$

Consequently (if it be thought necessary to adhere to integers throughout), for values of $n$ not exceeding 4, the old method would be probably the more expeditious.

## Addendum.

### *On a method of finding Superior and Inferior Limits to the real Roots of any Algebraical Equation.*

The theory above considered has incidentally led me to the discovery of a new and very remarkable method for finding superior and inferior limits to the real roots of any algebraical equation. Suppose in general that

$$\frac{N}{D}=\frac{1}{q_1+}\,\frac{1}{q_2+}\,\frac{1}{q_3+}\cdots+\frac{1}{q_n};$$

then it is easily seen that

$$D=M_1M_2M_3\ldots M_n,$$

where

$$M_1=q_1,\ M_2=q_2+\frac{1}{q_1},\ M_3=q_3+\frac{1}{M_2}\cdots M_n=q_n+\frac{1}{M_{n-1}}.$$

In general let any numerical quantity within brackets be used to denote its *positive* numerical value; so that, for instance, whether $q=\pm 3$, $(q)$ will equally denote $+3$.

And now suppose that neither $q_1$ nor $q_n$, the first or last of the quotients, lies between $+1$ and $-1$, and that no one of the intermediate quotients $q_2, q_3 \ldots q_{n-1}$ lies between $+2$ and $-2$; so that, in other words,

$$(q_1)>1,\ (q_2)>2,\ (q_3)>2\ldots(q_{n-1})>2,\ \ (q_n)>1;$$

then, I say, that $M_1, M_2, M_3 \ldots M_n$ will have the same signs as $q_1, q_2, q_3 \cdots q_n$ respectively; for

$$M_1=q_1,$$

therefore

$$(M_1)>1;$$

but

$$M_2=q_2+\frac{1}{M_1},$$

therefore $$(M_2) = (q_2) \pm \left(\frac{1}{M_1}\right) > 2 \pm 1;$$
therefore
$$M_2 \text{ has the same sign as } q_2, \text{ and } also\ (M_2) > 1;$$
therefore in like manner,
$$(M_3) \text{ has the same sign as } q_3, \text{ and } also\ (M_3) > 1;$$
therefore in like manner,
$$(M_4) \text{ has the same sign as } q_4, \text{ and } also\ (M_4) > 1;$$
and so on until we come to $M_{n-1}$, and we shall find
$$M_{n-1} \text{ of the same sign as } q_{n-1}, \text{ and } also\ (M_{n-1}) > 1.$$
Finally,
$$M_n = q_n \pm \frac{1}{M_{n-1}},$$
where $(q_n) > 1$ and $\left(\frac{1}{M_{n-1}}\right) < 1$, therefore
$$M_n \text{ has the same sign as } q_n;$$
but we cannot say (nor is there any occasion to say) that $(M_n) > 1$; therefore
$$D = M_1 M_2 M_3 \ldots M_n \text{ has the same sign as } q_1 q_2 q_3 \ldots q_n.$$

Now let $fx$ be any given function of $x$ of the $n$th degree, and $\phi x$ any *assumed* function whatever of $x$ of the $(n-1)$th degree, and let
$$\frac{\phi x}{fx} = \frac{1}{q_1 +} \frac{1}{q_2 +} \frac{1}{q_3 +} \ldots \frac{1}{q_n},$$
where $q_1, q_2, q_3 \ldots q_n$ are now supposed to be linear functions of $x$, which, except for *special relations* between $f$ and $\phi$, will always exist, and can be found by the ordinary process of successive division.

Write down the $n$ pairs of equations,
$$u_1 = q_1 + 1 = 0, \quad u_2 = q_2 + 2 = 0, \quad u_3 = q_3 + 2 = 0 \ldots u_n = q_n + 1 = 0,$$
$$u'_1 = q_1 - 1 = 0, \quad u'_2 = q_2 - 2 = 0, \quad u'_3 = q_3 - 2 = 0 \ldots u'_n = q_n - 1 = 0.$$

If the greatest of the values of $x$ determined from these $2n$ equations be called $L$, and the least of these values be called $\Lambda$, it may easily be made out that between $+\infty$ and $L$, each of the quantities $q_1, q_2, q_3 \ldots q_n$ will remain unaltered in sign; and between $-\infty$ and $\Lambda$ also the same invariability of sign obtains; and, moreover, between $+\infty$ and $L$, and between $\Lambda$ and $-\infty$, $(q_1), (q_2) \ldots (q_{n-1}), (q_n)$ will be respectively greater than 1, 2 … 2, 1. Consequently, by virtue of the preceding theorem, between $+\infty$ and $L$, and between $\Lambda$ and $-\infty$, $D$ will always retain the same sign as $q_1 q_2 q_3 \ldots q_n$, and therefore no root of $fx$ will be contained within either such interval. And hence $fx$, which is manifestly identical with $D$ (the denominator of the continued fraction last above written), affected with a certain constant factor, will retain an invariable sign within each such interval respectively. Hence, then, the following rule.

Calling $q_1, q_2, q_3 \dots q_n$ respectively

$$a_1x - b_1,\ a_2x - b_2,\ a_3x - b_3 \dots a_nx - b_n,$$

if we form the $2n$ quantities

$$\frac{b_1 \pm 1}{a_1},\ \frac{b_2 \pm 2}{a_2},\ \frac{b_3 \pm 2}{a_3} \dots \frac{b_{n-1} \pm 2}{a_{n-1}},\ \frac{b_n \pm 1}{a_n},$$

the greatest of these will be a superior limit, and the least of them an inferior limit to the roots of $fx$.

The values of these fractions will depend upon the form of the assumed subsidiary function $\phi$. Hence, then, arises a most curious question for future discussion—*to wit*, to discover whether in any case the subsidiary function can be so assumed as that the superior limit can be brought to coincide with the greatest, or the inferior limit with the least real root, supposing that there are any real roots. I believe that it will be found that this is always impossible to be done. Then, again, if all the roots are imaginary, can *inconsistent limits* (evincing this imaginariness) be obtained by giving different forms to the subsidiary function, which would be the case if we could find that the superior limit brought out by one form were less than the inferior limit brought out by another, or the inferior limit brought out by one form greater than the superior brought out by another? If, as I suspect, this also can never be done, then the general question remains to determine for all cases the form to be given to the subsidiary function, which will make the interval between either limit and its nearest root, or between the two limits themselves, a minimum. Thus, it appears to me, a fine field of research is thrown open to those who are interested in the theory of maxima minimorum, and minima maximorum, and one likely to lead to unexpected and important discoveries [cf. p. 533 above, and the Author's footnote, p. 495].

It may be asked how is the above rule to be applied if any of the leading coefficients in $\phi x$, or of the successive residues of $fx$ and $\phi x$ vanish; in which case, instead of the coefficients being linear, some of them will be, as in fact all might be, polynomial functions of $x$. The rule, it may be proved, will still subsist.

Equating the first and last quotients each of them to $+1$ and to $-1$, and the intermediate ones to $+2$ and to $-2$, the greatest root of all the equations so formed continues to be a superior, and the least root an inferior limit to the roots of $fx$. Nor is it ever necessary, even in these special cases, actually to *solve* any of these equations; for evidently it will be sufficient to find a superior limit and an inferior limit to each of them, and adopt the greatest of the superior and the least of the inferior limits as the superior and inferior limits to the roots of the given equation. Thus, then, we should have to repeat upon the quotients increased and diminished by 1 or 2 (as the case may be), the same process as is supposed to be originally applied to $fx$, and thus by a continued process of *trituration* (since every new function so to be operated upon is of a lower degree than the original function) we must finally descend to linear equations exclusively.

It is interesting thus to see that there are no failing cases in the application of the rule, and that a solution of equations of a higher degree than the first is never necessary. But as a matter of fact, the chances are infinitely improbable (if $\phi x$ is chosen at random), of any of the quotients after the first ceasing to be linear; and the first is of course linear, provided that the degree of $\phi x$ is taken only one unit below that of $fx$.

In working with Sturm's theorem, a system of quotients is supplied ready to hand; and these quotients, by virtue of the rule given above, may be used to assign a superior and inferior limit in the first instance, before setting about to determine the distribution of the roots between these limits by aid either of these same quotients or of the residues. For the change of sign of the residues required by the Sturmian process will only affect the signs, and not the forms of the quotients; but in the application of the above rule for finding the limits, the sign of any quotient is evidently immaterial.