63.

ON THE NEW RULE FOR FINDING SUPERIOR AND INFERIOR LIMITS TO THE REAL ROOTS OF ANY ALGEBRAICAL EQUATION.

[*Philosophical Magazine*, VI. (1853), pp. 138—140.]

THE lemma accessory to the demonstration of the rule for finding limits to the roots of an equation, given in the addendum [p. 623 above] to my paper in the *Magazine* for this month, admits of two successive and large steps of generalization, in which the scope of the principal theorem will participate in an equal degree.

1. Whatever the signs may be of $q_1, q_2, q_3 \dots q_r$, the denominator of the continued fraction

$$\frac{1}{q_1+}\frac{1}{q_2+}\frac{1}{q_3}\dots\frac{1}{q_r}$$

will have the same sign as $q_1q_2q_3 \dots q_r$, provided that

$$[q_1] > \mu_1,\ [q_2] > \mu_2 + \frac{1}{\mu_1},\ [q_3] > \mu_3 + \frac{1}{\mu_2} \dots$$

$$\dots [q_{r-1}] > \mu_{r-1} + \frac{1}{\mu_{r-2}},\ [q_r] > \frac{1}{\mu_{r-1}},$$

where $\mu_1, \mu_2 \dots \mu_{r-1}$ signify any *positive* quantities whatsoever; in the particular case where $\mu_1 = \mu_2 = \mu_3 = \dots = \mu_{r-1} = 1$, we fall back upon the lemma as originally stated.

2. But the lemma admits of another modification, which will in general impose far less stringent limits upon the arithmetical values of the series of q's.

Let all the possible sequences of q's be taken which present only variations of sign; for example if the entire series be q_1, q_2, q_3, q_4, and the corresponding algebraical signs are $+ - - +$, we shall have the two sequences q_1, q_2; q_3, q_4. If the entire series be $q_1, q_2, q_3 \dots q_{15}$, and the signs be

$$- - - + - + + + + - + + + + -,$$

then the sequences to be taken will be

$$q_3, q_4, q_5, q_6; \; q_9, q_{10}, q_{11}; \; q_{14}, q_{15},$$

and so in general.

Suppose, now, that $q_{\rho+1}, q_{\rho+2} \ldots q_{\rho+i}$ are the terms of any one such sequence. Then, provided that

$$[q_{\rho+1}] > \mu_1, \; [q_{\rho+2}] > \mu_2 + \frac{1}{\mu_1} \ldots q_{\rho+i-1} > \mu_{i-1} + \frac{1}{\mu_{i-2}},$$

and
$$q_{\rho+i} > \frac{1}{\mu_{i-1}},$$

(it being understood that the values of $\mu_1, \mu_2 \ldots \mu_{i-1}$ are perfectly arbitrary, except being subject to the condition of being all positive, and that there are as many distinct and independent systems of such values as there are sequences of variations of sign), it will continue to be true (and capable of being demonstrated to be so by precisely the same reasoning as was applied to the demonstration of the lemma in its original form) that the denominator of $\frac{1}{q_1 +} \frac{1}{q_2 +} \ldots \frac{1}{q_r}$ will have the same sign as the product $q_1 q_2 q_3 \ldots q_r$. It will be observed that, as regards the *residual* quotients not comprised in any sequence, their values are absolutely unaffected by any condition whatever. As a direct consequence from this lemma, we derive the following greatly improved *Theorem* for the discovery of the limits.

Let, as before, $fx = 0$ be any given algebraical equation; ϕx any assumed arbitrary function of x of an inferior degree to that of fx; and let

$$\frac{\phi x}{fx} = \frac{1}{X_1 +} \frac{1}{X_2 +} \frac{1}{X_3 +} \cdots \frac{1}{X_r};$$

let the leading coefficients of $X_1, X_2, X_3 \ldots X_r$ be $q_1, q_2, q_3 \ldots q_r$, and let this latter series be divided into sequences of variations and residual terms not comprised in any such sequence, as explained above. Let the X's corresponding to the residual terms be called

$$P_1, P_2 \ldots P_\omega,$$

and let the successive sets of X's corresponding to the sequences be called respectively

$$V_1, \; V_2 \ldots V_\rho,$$
$$V_1', \; V_2' \ldots V'_{\rho'},$$
$$V_1'', \; V_2'' \ldots V''_{\rho''},$$
$$\cdots\cdots\cdots\cdots$$
$$(V_1), (V_2) \ldots (V_{(\rho)}).$$

And let

$$X = P_1 P_2 \ldots P_\omega$$
$$\times (V_1^2 - c_1^2)(V_2^2 - c_2^2) \ldots (V_\rho^2 - c_\rho^2)$$
$$\times (V_1'^2 - c_1'^2)(V_2'^2 - c_2'^2) \ldots (V'_{\rho'}{}^2 - c'_{\rho'}{}^2)$$
$$\&c. \qquad \&c.$$
$$\times \{(V_1)^2 - (c_1)^2\}\{(V_2)^2 - (c_2)^2\} \ldots \{(V_{(\rho)})^2 - (c_{(\rho)})^2\},$$

where, in general, any system of values

$$c_1, c_2, c_3 \ldots c_{\rho-1}, c_\rho,$$

represents

$$\mu_1, \mu_2 + \frac{1}{\mu_1} \ldots \mu_{\rho-1} + \frac{1}{\mu_{\rho-2}}, \frac{1}{\mu_{\rho-1}}.$$

Then the largest root of $X = 0$ is a superior limit, and the smallest root of $X = 0$ is an inferior limit to the real roots of $fx = 0$; and if $X = 0$ has no real roots, neither will $fx = 0$ have any. For the complete demonstration and some further developments of this theorem see the forthcoming number of Terquem's *Nouvelles Annales* for the present month*.

[* p. 423 and p. 424 above.]