65.

THE ALGEBRAICAL THEORY OF THE SECULAR-INEQUALITY DETERMINANTIVE EQUATION GENERALIZED.

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ART. 1. Let

$$X_1 = ax + \alpha, \quad X_2 = \begin{bmatrix} ax+\alpha, & bx+\beta \\ bx+\beta, & cx+\gamma \end{bmatrix}, \quad X_3 = \begin{bmatrix} ax+\alpha, & bx+\beta, & dx+\delta \\ bx+\beta, & cx+\gamma, & ex+\epsilon \\ dx+\delta, & ex+\epsilon, & fx+\phi \end{bmatrix} \text{ \&c.,}$$

and let the first coefficients of X_1, X_2, X_3, &c. have all the same sign; then I say that the roots of any such function as X_i will be all real, and will lie respectively in the intervals comprised between $+\infty$, the successive descending roots of X_{i-1} and $-\infty$. When $a=1$, $c=1$, $f=1$, &c., and $b=0$, $d=0$, $e=0$, &c., $X_i=0$ becomes the well-known secular-inequality equation.

Demonstration. For greater simplicity, let all the first coefficients be taken positive, and suppose the theorem proved up to i, it will be true for $i+1$. For by a well-known property of symmetrical determinants, when $X_i=0$, X_{i-1} and X_{i+1} will have contrary signs. Let the roots of X_{i-1} be

$$h_1, \; h_2 \ldots h_{i-1},$$

and the roots of X_i,

$$k_1, \; k_2 \ldots k_{i-1}, \; k_i.$$

When $x=k_1$, which is greater than h_1, the greatest root of X_{i-1} will be positive; when $x=k_2$, which lies between the first and second roots of X_{i-1}, X_{i-1} will be negative; and so on, X_{i-1} alternately becoming positive and negative as we pass from root to root of X_i.

Hence X_{i+1}, which is positive when $x=\infty$, becomes negative when $x=k_1$, positive again when $x=k_2$, and so alternately; being finally, when $x=k_i$, positive or negative, and when $x=-\infty$, negative or positive, according as i is even or odd. Hence X_{i+1}, which changes sign $i+1$ times between $+\infty$ and $-\infty$, must have all its roots real, and lying severally in the intervals included between $+\infty$, the successive roots of X_i and $-\infty$. Hence if the theorem be true for $i-1$ and i, it is true for all numbers above i; but if we take

$$ax+\alpha \quad \text{and} \quad \begin{bmatrix} ax+\alpha, & bx+\beta \\ bx+\beta, & cx+\gamma \end{bmatrix}$$

the latter is $(ax + \alpha)(cx + \gamma) - (bx + \beta)^2$, which is positive for $x = \infty$, negative for $ax + \alpha = 0$, and positive for $x = -\infty$. Hence the theorem is true for X_1 and X_2, and therefore universally.

In the above demonstration it was supposed that the leading coefficients are all positive; but the demonstration will be precisely the same, *mutatis mutandis,* if they are all negative.

Art. 2. And *much more generally* it may be shown, in like manner, that if the successions of signs, in the series consisting of the sign + followed by the signs of the principal coefficients in $X_1, X_2 \ldots X_{m+n}$, consist of m variations and n continuations, the number of real roots of the equation $X_{m+n} = 0$ will be *at least* as great as the positive value of the difference between m and n. This theorem, moreover, remains true if X_1, X_2, X_3, &c. be formed from a symmetrical matrix, in which the terms, instead of being linear functions of x, are any odd-degreed rational integral functions of x, or fractional functions of which the numerators (when rendered prime to their denominators) are odd-degreed functions of x. My friend M. Borchardt, who has so beautifully effected the decomposition of my formulæ for the Sturmian criteria of reality into the sums of squares for the secular-inequality form of the equation, may now, if he pleases, tax his ingenuity to effect a similar decomposition for the general case supposed in Art. 1*.

Art. 3. It is obvious that, in applying the theorem contained in Arts. 1 and 2, it is indifferent whether we look to the signs of the successive determinants a; $\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}$; &c., or to those of α; $\begin{vmatrix} \alpha, & \beta \\ \beta, & \gamma \end{vmatrix}$; &c.; or, more generally, to those of $a + \alpha\theta$; $\begin{vmatrix} a + \alpha\theta, & b + \beta\theta \\ b + \beta\theta, & c + \gamma\theta \end{vmatrix}$; &c., θ being any arbitrary but real quantity. Conversely we obtain the remarkable theorem, that *when any homogeneous quadratic function, whose coefficients are linear functions of θ,*

* So, too, my own more simple method for proving the omni-reality of the roots of the secular-inequality equation, August 1852, [p. 364 above], ought to be capable of being extended to the general form in Art. 1, that is we ought to be able to prove that the equation whose roots are the squares of the roots of $X_i = 0$ will have all its coefficients alternately negative and positive. If we take for example $i = 2$, the equation to the squares of the roots becomes

$$(ac - b^2)^2 x^2 - \{(a\gamma + c\alpha - 2b\beta)^2 + 2(b^2 - ac)(\alpha\gamma - \beta^2)\} x + (\alpha\gamma - \beta^2)^2 = 0;$$

and we have to prove that the coefficient of $-x$ in this equation is essentially positive when $ac - b^2$ is positive: this may be shown by various modes of decomposition; amongst others, by writing the coefficient in question under the form

$$\frac{1}{c^2}\{(c^2\alpha + \gamma b^2 - 2bc\beta)^2 + \gamma^2(ac - b^2)^2 + 2(b\gamma - c\beta)^2(ac - b^2)\}.$$

In general, if L is essentially positive when L_1, $L_2 \ldots L_i$ are positive, then, discarding all artifices of calculation, this must be capable of being proved by virtue of an identity of the form

$$L = \Sigma m^2 + \Sigma m_1^2 L_1 + \Sigma m_2^2 L_2 + \ldots + \Sigma m_i^2 L_i.$$

is linearly converted by real substitutions into a sum of positive and negative squares, the greatest difference for any value of θ between the number of positive and the number of negative squares has for its limit the number of real roots of θ in the Discriminant (otherwise called the Determinant) of the given function. The theorem actually demonstrated above teaches only this much, namely that the maximum difference in number between the two species of squares (which depends only on the value given to θ) cannot exceed the number of real roots in the discriminant; it admits, however, of an easy proof that this maximum difference is equal to the number of real roots, so that the one number is, in the strict sense of the word, an *exact* limit to the other.

Art. 4. I was led to the theorem, as given in Art. 1, by having to consider the following curious and important question.

"*Given i linear functions of x, say $X_1, X_2 \ldots X_i$, to find the $i-1$ positive quantities, say $\mu_1, \mu_2 \ldots \mu_{i-1}$, which shall give the least value to the greatest root, or the greatest value to the least root, of the equation*

$$(X_1^2-\mu_1^2)\left\{X_2^2-\left(\mu_2+\frac{1}{\mu_1}\right)^2\right\}\left\{X_3^2-\left(\mu_3+\frac{1}{\mu_2}\right)^2\right\}\ldots\left\{X_i^2-\left(\frac{1}{\mu_{i-1}}\right)^2\right\}=0."$$

The theorem in Art. 1 enables me easily to demonstrate, that if we take $X_1', X_2', X_3' \ldots X_i'$ identical with

$$\sqrt{1}\,.\,X_1,\ \sqrt{1}\,.\,X_2 \ldots \sqrt{1}\,.\,X_i,$$

the sign of the square root being selected in each case so that the coefficients of x in $X_1', X_2' \ldots X_i'$ shall have all the same sign, then the least value of the greatest root, and the greatest value of the least root, of the given equation will be respectively the greatest and least finite roots of the equation

$$X_1' - \frac{1}{X_2' -}\,\frac{1}{X_3'}\cdots\frac{1}{X_i'} = 0^*;$$

the two systems of values of $\mu_1, \mu_2 \ldots \mu_{i-1}$ required being the two systems of values of

$$X_1',\ X_2' - \frac{1}{X_1'},\ X_3' - \frac{1}{X_2' -}\,\frac{1}{X_1'} \cdots X'_{i-1} - \frac{1}{X'_{i-2} -}\,\frac{1}{X'_{i-3}}\cdots\frac{1}{X_1'},$$

corresponding respectively to these two values of x.

And it is by means of this solution that the statement of the rule for finding the superior and inferior limits to the real roots of an algebraical equation made in the last August Number of the *Magazine*, is capable of being converted into the statement contained in the third observation on the same rule in the present Number [p. 631 above].

* The finite roots of this are the same as those of

$$X_i' - \frac{1}{X'_{i-1} -}\,\frac{1}{X'_{i-2} -}\cdots\frac{1}{X_1'} = 0.$$