EDITOR'S NOTE ON SYLVESTER'S THEOREMS FOR DETERMINANTS IN THIS VOLUME.

In Sylvester's paper No. 37, p. 241 above, beside the errors noticed by Sylvester himself, pp. 251 and 401 of this volume, the substitution of $b_{\phi_1}\ldots b_{\phi_r}$ for $a_{\phi_1}\ldots a_{\phi_r}$ in line 22 of p. 244 and the substitution of $a_{\theta_1}\ldots a_{\theta_r}$ for $b_{\theta_1}\ldots b_{\theta_r}$ in line 8 of p. 245, there is the more fundamental error that in formula (2), p. 244, and the formula at the foot of p. 247 the suffixes of the b's should be $k_1\ldots k_r$ and $l_1\ldots l_r$, and the suffixes of the a's should be $\theta_1\ldots\theta_r$ and $\phi_1\ldots\phi_r$. It may be a convenience to the reader to have at hand another view of Sylvester's three main theorems on determinants in this volume (pp. 247, 253, 249).

1. A matrix of type (m, n) is an object of calculation depending on mn numbers which we suppose arranged as a rectangle of m rows and n columns. By the product $(a)(b)$ of two matrices (a), (b) of respective types (n_1, m), (m, n_2) is meant the matrix of type (n_1, n_2) which has for its (p, q)th element, that is the q-th element of its p-th row, the number

$$a_{p1}b_{1q}+\ldots+a_{pm}b_{mq},$$

where a_{pr}, b_{rq} are respectively the (p, r)th and (r, q)th elements of (a) and (b).

If i denote a particular one of the $\binom{n_1}{r}$ possible selections of r numbers from $1, 2, \ldots, n_1$, say $i_1\ldots i_r$, and j denote a particular one of the $\binom{n_2}{s}$ possible selections of s numbers from $1, 2, \ldots, n_2$, say $j_1\ldots j_s$, we may pick out from the product matrix $(a)(b)$ a minor matrix of r rows and s columns consisting of the elements of this common to the rows $i_1\ldots i_r$ and the columns $j_1\ldots j_s$; this is clearly given by

$$((a)(b))_{ij}=\begin{pmatrix} a_{i_1 1} & \cdots & a_{i_1 m} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots} \\ a_{i_r 1} & \cdots & a_{i_r m}\end{pmatrix}\begin{pmatrix} b_{1j_1} & \cdots & b_{1j_s} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots} \\ b_{mj_1} & \cdots & b_{mj_s}\end{pmatrix}=(a)_i\,(b)^j,$$

where $(a)_i$ is the matrix of type (r, m) constituted by the i-rows of (a), and $(b)^j$ the matrix of type (m, s) constituted by the j-columns of (b). When $s=r$ this matrix is square and, if k denotes a selection of r numbers from $1, 2, \ldots, m$, its determinant is given by

$$|((a)(b))_{ij}|=\sum_k |(a)_i^k|\,|(b)_k^j|,$$

where $|(a)_i^k|$, which we may denote by $|(a)_{ik}|$, denotes the determinant of the minor of (a)

formed with its i-rows and its k-columns, and $|(b)_k{}^j|$ or $|(b)_{kj}|$ denotes the determinant of the minor formed with the k-rows and the j-columns of (b), and the summation extends to the possible $\binom{m}{r}$ significations of k. Similarly if (a), (b), (c) be matrices of respective types (n_1, n), (n, m), (m, n_2), the minor with the rows i and the columns j of the product matrix $(a)(b)(c)$ of type (n_1, n_2) is given by

$$((a)(b)(c))_{ij} = ((a)(b))_i (c)^j = (a)_i (b)(c)^j,$$

and when $s=r$, its determinant is

$$|((a)(b)(c))_{ij}| = \sum_k |((a)(b))_{ik}|\,|(c)_{kj}|$$

$$= \sum_k \sum_h |(a)_{ih}|\,|(b)_{hk}|\,|(c)_{kj}|,$$

where k is as before and h denotes a selection of r numbers from 1, 2, ..., n, the summation extending to the $\binom{m}{r}\binom{n}{r}$ possible significations of k, h.

2. Hence the theorem of Sylvester on the minor determinants of linearly equivalent quadratic functions, pp. 244, 247 above. For if by the substitution

$$x_1 = \mu_{11} y_1 + \dots + \mu_{1n} y_n, \ \dots, \ x_n = \mu_{n1} y_1 + \dots + \mu_{nn} y_n,$$

the quadratic form $a_{11} x_1{}^2 + \dots + 2a_{12} x_1 x_2 + \dots$ become $b_{11} y_1{}^2 + \dots + 2b_{12} y_1 y_2 + \dots$, we at once find

$$b_{pq} = \sum_{s=1}^{n} \mu_{sp} (a_{s1} \mu_{1q} + \dots + a_{st} \mu_{tq} + \dots + a_{sn} \mu_{nq}),$$

so that the matrix of the new form is given by

$$(b) = (\bar{\mu})(a)(\mu),$$

where a_{pq}, μ_{pq}, μ_{qp}, b_{pq} are the (p, q)th elements respectively of the matrices (a), (μ), $(\bar{\mu})$, (b), which are all of type (n, n). Supposing the numbers n_1, n, m, n_2 of § 1 all equal to n, the determinant of the (i, j)th minor of order r in (b) is

$$\sum_k \sum_h |(\bar{\mu})_{ih}|\,|(a)_{hk}|\,|(\mu)_{kj}|,$$

or

$$\sum_k \sum_h |(a)_{hk}|\,|(\mu)_{hi}|\,|(\mu)_{kj}|,$$

this being the result which in the notation of Sylvester would be written

$$\sum_h \sum_k \begin{pmatrix} a_{h_1} & a_{h_2} & \dots & a_{h_r} \\ a_{k_1} & a_{k_2} & \dots & a_{k_r} \end{pmatrix} \begin{pmatrix} \mu_{h_1} & \mu_{h_2} & \dots & \mu_{h_r} \\ \mu_{i_1} & \mu_{i_2} & \dots & \mu_{i_r} \end{pmatrix} \begin{pmatrix} \mu_{k_1} & \mu_{k_2} & \dots & \mu_{k_r} \\ \mu_{j_1} & \mu_{j_2} & \dots & \mu_{j_r} \end{pmatrix},$$

the first row giving the rows used to form any minor determinant. It will be noticed that the columns of the matrix (μ) which come into consideration are those of the same enumeration as the rows and columns of the minor of the matrix (b) which is to be expressed; this is contrary to Sylvester's formula of p. 247 above.

3. When the product of two square matrices (a), (b), each of type (n, n), is the so-called unit matrix, in which every element is zero save those in the diagonal which are each unity, the matrices are called inverse; and we have $(a)(b) = 1 = (b)(a)$. Denoting by α_{ij} the determinant of a minor matrix of type (r, r) formed with rows $i_1 \dots i_r$ and columns $j_1 \dots j_r$ from (a), and by α'_{ij} the determinant of the complementary matrix of type $(n-r, n-r)$, we have, if $A = |(a)|$, $\mu = \binom{n}{r}$, by Laplace's rule for the expansion of a determinant

$$\alpha_{i1} \alpha'_{j1} + \dots + \alpha_{i\mu} \alpha'_{j\mu} = A, \text{ or, } 0,$$

according as $i=j$ or $i \neq j$. Thus the two matrices of type (μ, μ), in which the (i, j)th element of the first is α_{ij}, and the (i, j)th element of the second is $\frac{\alpha'_{ji}}{A}$, are inverse to one another, so that we have

$$(\alpha)\left(\frac{\bar{\alpha}'}{A}\right) = \left(\frac{\bar{\alpha}'}{A}\right)(\alpha) = \left(\frac{\alpha'}{A}\right)(\bar{\alpha}) = (\bar{\alpha})\left(\frac{\alpha'}{A}\right) = 1,$$

where the bar above the symbol for a matrix indicates the transposed matrix differing from the original in having its first, second, ... rows those respectively which were the first, second, ... columns of the original. From this equation it is easy to prove that the determinant of the matrix (a) is the $\frac{\mu r}{n}$th power of A. If (b) be inverse to (a), and β_{ij} be the determinant of type (r, r) formed from (b) as was a_{ij} from (a), it follows by considering (see § 1) the determinant of the product of the matrix $(a)_i$ of type (r, n) formed by the i-rows of (a) and the matrix $(b)^j$ of type (n, r) formed by the j-columns of (b), that

$$\alpha_{i1}\beta_{1j} + \alpha_{i2}\beta_{2j} + \ldots + \alpha_{i\mu}\beta_{\mu j} = 1, \text{ or, } 0,$$

according as $i = j$ or $i \neq j$; hence the matrices (α) and (β) are inverse; and thus, by the above

$$\beta_{ij} = \frac{\alpha'_{ji}}{A},$$

or in words, any minor determinant of the inverse of a given matrix is equal to the complementary determinant formed from the transposed of the original matrix divided by the determinant of the original matrix. In particular this gives the elements of the inverse matrix expressed by minors of the original.

If (a), (b) be any two matrices of type (n, n) we can form a matrix of type (n, n) by replacing the i-th selection of r rows in (a), by the j-th selection of r rows of (b); this matrix being called $(a, b)_{ij}$ and its determinant $|a, b|_{ij}$, we have, by Laplace's rule

$$|a, b|_{ij} = \alpha'_{i1}\beta_{j1} + \alpha'_{i2}\beta_{j2} + \ldots + \alpha'_{i\mu}\beta_{j\mu};$$

hence the matrix, of type (μ, μ), of which the (i, j)th element is $\frac{|a, b|_{ij}}{A}$, is given by

$$\left(\frac{|a, b|}{A}\right) = \left(\frac{\alpha'}{A}\right)(\bar{\beta});$$

thus, by means of $(\bar{\beta})\left(\frac{\beta'}{B}\right) = 1$, $\left(\frac{\alpha'}{A}\right)(\bar{\alpha}) = 1$, we have

$$\left(\frac{|a, b|}{A}\right)\left(\frac{|b, a|}{B}\right) = \left(\frac{\alpha'}{A}\right)(\bar{\beta})\left(\frac{\beta'}{B}\right)(\bar{\alpha}) = 1,$$

and the matrices

$$\left(\frac{|a, b|}{A}\right), \quad \left(\frac{|b, a|}{B}\right),$$

are inverse; this is Sylvester's theorem p. 253 above.

We remark, using 1 for the unit matrix, the relations, where (c) is of type (n, n),

$$|a, 1|_{ij} = \alpha'_{ij}, \quad |1, a|_{ij} = \alpha_{ji}, \quad (a, b)_{ij}(c) = (ac, bc)_{ij},$$

of which the last gives, if (b), $= (a)^{-1}$, be inverse to (a),

$$(a^{-1}, 1)_{ij}(a) = (1, a)_{ij}, \text{ and hence } \beta'_{ij} = \frac{\alpha_{ji}}{A},$$

as proved above.

4. Let $n > r > m$, and (a) be of type (n, n). A fixed minor M_m of type (m, m) from (a) determines a complementary minor of type $(n-m, n-m)$, say M_{n-m}. From the $n-m$ numbers, say $p_1 \ldots p_{n-m}$, enumerating the rows of M_{n-m}, make a selection $\theta_1 \ldots \theta_{r-m}$, and from the numbers, say $q_1 \ldots q_{n-m}$, enumerating the columns of M_{n-m} make a selection $\phi_1 \ldots \phi_{r-m}$; then form a minor M_r of (a), of type (r, r), whose rows are enumerated by those of M_m together with $\theta_1 \ldots \theta_{r-m}$, and columns by those of M_m together with $\phi_1 \ldots \phi_{r-m}$; let the rows and columns of (a) not now enumerated be given respectively by $\theta_1' \ldots \theta'_{n-r}$ and $\phi_1' \ldots \phi'_{n-r}$; let (b) be inverse to (a). Then the determinant of M_r is equal to the determinant formed from $(\bar{b})$ with the rows $\theta_1' \ldots \theta'_{n-r}$ and the columns $\phi_1' \ldots \phi'_{n-r}$, multiplied by A. Now suppose $\theta_1 \ldots \theta_{r-m}$ to become in turn all the

$\binom{n-m}{r-m}$ possible selections from $p_1 \dots p_{n-m}$, and similarly $\phi_1 \dots \phi_{r-m}$ all from $q_1 \dots q_{n-m}$; the determinants M_r so obtained form a matrix H of $\binom{n-m}{r-m}$ rows and columns, which is in fact a minor of the previously considered matrix (a). We wish to determine the determinant of this matrix H. Now the determinants $(\theta_1' \dots \theta'_{n-r}, \phi_1' \dots \phi'_{n-r})$ of $(\bar{b})$, complementary in indices to the matrices M_r, are minors of the matrix $(p_1 \dots p_{n-m}, q_1 \dots q_{n-m})$ of $(\bar{b})$, and the matrix of order $\binom{n-m}{r-m}$ formed from them has therefore for its determinant $\Delta_1{}^{\lambda}$, where Δ_1 is the determinant of this matrix $(p_1 \dots p_{n-m}, q_1 \dots q_{n-m})$ of $(\bar{b})$, and $\lambda = \binom{n-m-1}{r-m}$; hence, as $\Delta_1 = \frac{\Delta}{A}$, where Δ is the determinant of the fixed matrix M_m of (a), the determinant H, of order $\binom{n-m}{r-m}$, of the minors M_r of (a), is equal to

$$\left(\frac{\Delta}{A}\right)^{\lambda} A^{\mu} = \Delta^{\lambda} A^{\sigma},$$

where $\mu = \binom{n-m}{r-m}$, $\sigma = \binom{n-m-1}{r-m-1}$. And this is Sylvester's theorem, p. 249 above.

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